Accumulation and Welfare[*]

Eduardo Dávila [†]   Abdelrahman Hassanein [‡]

August 2026

Abstract

This paper identifies the origins of welfare gains and losses in economies with accumulation technologies. Welfare and production efficiency gains from any perturbation originate through five components: cross-sectional and aggregate investment efficiency, cross-sectional capital use efficiency, and accumulation and production technology change, regardless of the institutional or market structure. In frictionless competitive economies, allocative components vanish, and a new accumulation Hulten's theorem values improvements in accumulation technologies by accumulation Domar weights: the value of capital stocks relative to consumption expenditures. In U.S. data for 1947–2024, the accumulation Domar sum exceeds the production Domar sum every year, indicating that innovations to accumulation technologies can be more valuable than innovations to production. An application to a credit crunch separates three sources of welfare losses — capital misallocation across producers, intertemporal-sharing, and redistribution between constrained and unconstrained individuals — from a perhaps unexpected gain: because collateral constraints induce overinvestment, a credit crunch shifts resources from investment toward consumption, where they are more valuable.

JEL Classification: D61, E22, E23, G31, O41

Keywords: welfare accounting, capital accumulation, production efficiency, accumulation Hulten's theorem

[*]We would like to thank Andy Abel, Susanto Basu, and Ellen McGrattan for helpful comments and discussions. We also would like to thank Andreas Schaab for valuable input in the early stages of this project. This material is based upon work supported by the National Science Foundation under Grant DGE-2237790.

[†]Yale University and NBER. Email: eduardo.davila@yale.edu

[‡]University of California, Berkeley. Email: ahassanein@berkeley.edu

1 Introduction

Capital accumulation determines how economies transform current resources into future consumption possibilities. Because capital is durable, resource constraints are linked across dates: a perturbation that modifies investment today changes the capital available tomorrow, which in turn determines the goods produced and the capital accumulated at every subsequent date. Assessing the welfare impact of changes in technologies, policies, or allocations in economies with capital is consequently far from trivial. It requires tracing how a perturbation propagates through production and accumulation over time.

In this paper, we systematically identify the origins of welfare gains and losses in economies with accumulation technologies. We consider a general dynamic environment with multiple goods and multiple capital goods: goods are contemporaneously produced using capital, while capital is created over time through accumulation technologies that use capital and investment. Our environment nests the workhorse models of capital accumulation, including the neoclassical growth model (Ramsey, 1928; Cass, 1965; Koopmans, 1965), investment-specific technological change (Greenwood, Hercowitz, and Krusell, 1997), two-sector growth models (Uzawa, 1965; Lucas, 1988; Rebelo, 1991), investment networks (Horvath, 2000; vom Lehn and Winberry, 2022), and credit cycles (Kiyotaki and Moore, 1997).[1]

Origins of Production Efficiency Gains. Our first main result, Theorem 1, shows that in this environment welfare gains coincide with production efficiency gains, and attributes them to five components: (i) cross-sectional investment efficiency, which measures the gains from reallocating investment across capital goods; (ii) aggregate investment efficiency, which measures the gains from adjusting the share of the aggregate supply of goods devoted to investment rather than consumption; (iii) cross-sectional capital use efficiency, which measures the gains from reallocating capital across producers; (iv) accumulation technology change; and (v) production technology change. The first three components capture allocative efficiency gains, since they are associated with the reallocation of resources across the economy; the last two capture technical efficiency gains from primitive changes in technologies. A defining feature of our approach is that the

[1]For clarity, the main text studies a deterministic economy with a single (representative) individual; the Online Appendix extends our results to heterogeneous individuals, elastically supplied factors, intermediate inputs, and uncertainty.

decomposition is exclusively based on preferences, technologies, and resource constraints: it makes no reference to prices, budget constraints, or equilibrium notions. It therefore identifies the origins of the welfare gains from any perturbation of any feasible allocation — efficient or not — under any institutional or market structure, and none of the results rely on optimality conditions.

The decomposition is built from two sets of objects. First, we identify propagation matrices — a capital inverse matrix along with goods and goods-capital inverse matrices — that characterize how changes in capital and investment at a given date translate into goods and capital at all subsequent dates. These are purely technological objects, defined at a given allocation without reference to prices or competitive notions. Second, we characterize welfare-relevant statistics — marginal social values and marginal welfare products — that combine the propagation matrices with preferences to translate changes in allocations and technologies into welfare.

Since the allocative components are systematically expressed in terms of changes in allocation shares, the decomposition separates gains due to reallocation — holding aggregate investment, goods supply, and capital use fixed — from gains due to changes in aggregates. We also present, to our knowledge, the first characterization of efficiency conditions in economies with capital and investment at this level of generality.

Technology Change in Frictionless Competitive Economies. Our second main result, Theorem 2, characterizes the decomposition in frictionless competitive economies. Since the competitive allocation is efficient, marginal reallocations of investment or capital have no welfare impact: all allocative efficiency components vanish, and production efficiency originates exclusively in primitive changes in technologies, valued at equilibrium prices — an additional unit of good j is worth its discounted goods price, and an additional unit of capital good z is worth its discounted capital price. Expressed in proportional terms, this result extends Hulten's theorem to accumulation technologies: while the welfare gain from a one-percent improvement in a production technology equals its production Domar weight — the value of its gross output relative to nominal consumption expenditures (Hulten, 1978) — the welfare gain from a one-percent improvement in an accumulation technology equals a new statistic, its accumulation Domar weight, defined as the value of the corresponding capital stock relative to nominal consumption expenditures.

The value of assets emerges as the key object for the welfare assessment of technological change. The date-t price of capital, q_t^z, is sufficient to value improvements in accumulation technologies, even though the additional capital generates goods and further capital at every subsequent date. The reason is that the price of installed capital already capitalizes the entire sequence of future propagation, so a single date-t price suffices. This logic also determines how welfare gains depend on the persistence of a technology change: a transitory improvement in a production technology raises output at a single date, while a transitory improvement in an accumulation technology delivers durable capital whose services decay only at the rate of depreciation — yet its welfare gain is still summarized by a single accumulation Domar weight. Permanent changes in accumulation technologies can consequently have large welfare effects, as they correspond to a discounted sum of capital prices, not of capital rents.

Moreover, we show that there exists an unambiguous relation between production and accumulation Domar sums: at every date, the accumulation Domar sum weakly exceeds the production Domar sum, which weakly exceeds one. The proof relies on finding a chain of valuations: consumption expenditures fall short of gross output because part of output is invested; gross output equals capital rentals in our baseline environment, in which capital goods are the only production inputs; and rentals fall short of the value of capital because installed capital carries continuation value into the future. When production technologies also use intermediate inputs and additional factors, the ordering can in principle reverse, so whether across-the-board improvements in accumulation technologies are more valuable than improvements in production technologies is an empirical question — the subject of our first application.

Measuring Accumulation and Production Domar Weights. Our first application measures production and accumulation Domar weights for the postwar U.S. economy. Using Bureau of Economic Analysis (BEA) input-output and fixed-asset data for 20 industries between 1947 and 2024, we find that the accumulation Domar sum exceeds the production Domar sum in every year of the sample: the former fluctuates between 3.4 and 4.4, while the latter remains between 2.0 and 2.5, so their ratio ranges from 1.4 to 1.9. A permanent one-percent Hicks-neutral improvement in all technologies yields welfare gains of 6%, with roughly 61% originating in accumulation technologies. Domar weights are concentrated: Real Estate, Public Administration, and Manufacturing jointly account for almost two thirds of the total, and, across capital-good categories, Structures absorb 79%

of the accumulation total. An interpretation of these magnitudes is that technological innovations to accumulation can be more valuable than technological innovations to production.

Origins of Welfare Losses in a Credit Crunch. Our second application identifies the origins of the welfare losses induced by a credit crunch. We study a two-sector economy in which manufacturing operates subject to a collateral constraint — a two-individual variant of Banerjee and Moll (2010), with the credit-crunch experiment of Buera and Moll (2015) — so that a change in the tightness of the borrowing constraint operates simultaneously through several margins. Our decomposition separates the welfare losses into capital misallocation across producers, aggregate capital accumulation, and transfers between constrained and unconstrained individuals. A marginal credit crunch reduces utilitarian welfare at every level of credit-market tightness: capital misallocation and transfers between constrained and unconstrained individuals generate losses that grow as the constraint tightens. Perhaps unexpectedly, the aggregate investment margin generates a welfare gain: collateral constraints induce overinvestment — every unit of wealth relaxes future constraints — so a credit crunch shifts resources from investment toward consumption, where their marginal social value is higher. This is the type of phenomenon that our approach is designed to uncover.

Related Literature. Our results connect to classical studies of optimal capital accumulation — Ramsey (1928), Cass (1965), Koopmans (1965), and Uzawa (1969) — and to the characterizations of dynamic efficiency in Cass (1972) and Abel, Mankiw, Summers, and Zeckhauser (1989). While this work characterizes efficient accumulation paths in aggregative economies, our decomposition identifies the origins of the welfare gains from any perturbation of any allocation — efficient or not — in economies with multiple goods, factors, and capital goods. Our results also connect to neoclassical investment theory and q-theory, as in Jorgenson (1963), Tobin (1969), Abel (1981), Hayashi (1982), and Abel and Eberly (1994). In frictionless competitive economies, we show that asset values embed the welfare-relevant statistics of our decomposition, and we use the shadow price of installed capital to value improvements in accumulation technologies rather than to characterize a firm's investment rule.

The characterization of welfare gains in frictionless competitive economies relates our results to Hulten's theorem (Hulten, 1978) and to the growth accounting literature that follows Solow (1957), including Hall (1990), Basu and Fernald (2002), and Baqaee and Farhi (2020). The welfare content of measured national aggregates in economies with accumulation is the subject of Kuznets (1941), Nordhaus and Tobin (1973), Weitzman (1976), and Hulten (1979), and, more recently, of Barro (2021) and Basu, Pascali, Schiantarelli, and Servén (2022). Relative to this work, the way in which we attribute welfare gains to sources (i) makes no assumptions about individual behavior, budget constraints, prices, or equilibrium notions, and (ii) decomposes welfare gains rather than changes in output or measured productivity. Our accumulation Hulten's theorem shows that changes in production technologies are valued by production Domar weights, while changes in accumulation technologies are valued by a new set of weights — accumulation Domar weights — that measure the value of capital stocks relative to nominal consumption expenditures. Business cycle accounting (Chari, Kehoe, and McGrattan, 2007) shares our goal of attributing observed outcomes to distinct margins, including an investment wedge, and Berger, Bocola, and Dovis (2023) show that heterogeneous-agent economies with imperfect risk sharing admit a representative-agent representation with wedges. These approaches account for output dynamics through the lens of a representative agent equilibrium model with prices, whereas we decompose welfare gains without taking a stance on the underlying market structure.

Since our decomposition places no restrictions on market structure or individual behavior, it applies directly to the workhorse macroeconomic models with capital: economies with investment-specific technological change (Greenwood, Hercowitz, and Krusell, 1997; Fisher, 2006), investment networks (vom Lehn and Winberry, 2022; Foerster, Hornstein, Sarte, and Watson, 2022), financial frictions (Kiyotaki and Moore, 1997; Brunnermeier and Sannikov, 2014), and capital misallocation (Restuccia and Rogerson, 2008; Hsieh and Klenow, 2009; Buera, Kaboski, and Shin, 2011; Moll, 2014; David and Venkateswaran, 2019).[2] Our second application illustrates this generality by tracing the welfare losses of a credit crunch back to their origins: while output-based measures attribute the losses of financial crises to capital misallocation, our decomposition separates the losses due to misallocating capital across producers from the effects of changing aggregate accumulation and of transferring resources between constrained and

[2]Work on sectoral shocks and comovement in economies with capital includes Horvath (2000), Foerster, Sarte, and Watson (2011), Atalay (2017), and Buera and Moll (2015).

unconstrained individuals.

Our results also relate to recent work on capital accumulation. Baqaee and Malmberg (2025) represent balanced-growth paths of models with capital as equivalent static economies in which capital services act as intermediate inputs; their analysis is positive and concerns comparisons across steady states, while ours is normative and values the entire sequence of allocations generated by transitory or permanent perturbations, including economies away from steady state and allocations that need not be competitive or efficient. Gaillard, Morrison, and Wangner (2026) derive a criterion for over- or under-saving in overlapping generations economies with household heterogeneity, extending the golden rule to account for uninsurable risk and shifts in the income distribution. Carvalho, Covarrubias, and Nuño (2026) quantify the welfare costs of disasters in dynamic production networks, in which a planner self-insures both by accumulating more capital and by reallocating capital across sectors — margins connected to the aggregate and cross-sectional investment efficiency terms of our decomposition.

Finally, this paper extends the welfare accounting framework of Dávila and Schaab (2025, 2023) to economies with accumulation technologies. This approach systematically identifies the origins of welfare gains in economies with rich heterogeneity, exclusively based on preferences, technologies, and resource constraints. In that paper, all goods are nonstorable, so production and valuation take place within each date. Here, we explicitly model accumulation technologies, which link resource constraints across dates. These links create three new allocative margins — aggregate investment efficiency, cross-sectional investment efficiency, and cross-sectional capital use efficiency — in addition to technological changes in accumulation.

2 Environment

We first introduce preferences, production and accumulation technologies, and resource constraints, and then define feasible allocations and perturbations.[3]

Preferences. We consider a deterministic economy with a single (representative) individual. Time is discrete, with dates indexed by $t \in \{0, \dots, T\}$, where $T \leq \infty$. The

[3]For clarity, the main text focuses on a deterministic economy with a single individual and only capital goods as production inputs. Section B of the Online Appendix shows how to accommodate heterogeneous individuals, elastically and inelastically supplied factors, intermediate inputs, and uncertainty.

individual consumes $J \geq 1$ perishable goods and services — *goods* for short — indexed by $j, \ell \in \mathcal{J} = \{1, \ldots, J\}$. There are also $Z \geq 1$ durable goods — *capital goods* or capital for short — indexed by $z \in \mathcal{Z} = \{1, \ldots, Z\}$. Goods are contemporaneously produced using capital goods as inputs. Capital goods are created using goods as inputs through investment.

Individual preferences are represented by

$$\text{(Preferences)} \qquad V = \sum_t \beta^t u_t \left(\left\{ c_t^j \right\}_{j \in \mathcal{J}} \right), \tag{1}$$

where c_t^j denotes final consumption of good j at date t, $\beta \in [0, 1)$ denotes the time discount factor, and $u_t(\cdot)$ corresponds to date-t instantaneous utility.

Production and Accumulation Technologies. Goods are produced using technologies that take capital goods as inputs. The production technology for good j at date t, denoted by $G_t^j(\cdot) \geq 0$, is given by

$$\text{(Production Technologies)} \qquad y_t^j = G_t^j \left(\left\{ k_t^{jz,d} \right\}_{z \in \mathcal{Z}}; \theta \right), \tag{2}$$

where y_t^j denotes the amount produced (output) of good j at date t, and $k_t^{jz,d}$ denotes the amount of capital good z used in the production of good j at date t (the superscript d stands for demand). We parametrize $G_t^j(\cdot; \theta)$ by θ to consider perturbations to production technologies, as described below.

Accumulation technologies govern the dynamics of capital: they specify the stock of capital available at date $t+1$ as a function of how capital is used at date t and of investment at that date. The accumulation technology for capital good z between dates t and $t+1$, denoted by $H_{t+1}^z(\cdot) \geq 0$, is given by

$$\text{(Accumulation Technologies)} \qquad k_{t+1}^z = H_{t+1}^z \left(\left\{ k_t^{jz,d} \right\}_{j \in \mathcal{J}}, \left\{ \iota_t^{zj} \right\}_{j \in \mathcal{J}}; \theta \right), \tag{3}$$

where k_{t+1}^z denotes the aggregate supply of capital good z at date $t+1$, and ι_t^{zj} denotes the amount of good j invested in capital good z at date t. We parametrize $H_{t+1}^z(\cdot; \theta)$ by θ to consider perturbations to accumulation technologies, as described below.

Resource Constraints. The resource constraint for good j at date t is

$$\text{(Resource Constraints: Goods)} \qquad y_t^j = c_t^j + \iota_t^j, \tag{4}$$

where $\iota_t^j = \sum_z \iota_t^{zj}$ denotes the aggregate investment of good j at date t.

The resource constraint for capital good z at date t is

$$\text{(Resource Constraints: Capital Goods)} \qquad k_t^z = k_t^{z,d}, \tag{5}$$

where $k_t^{z,d} = \sum_j k_t^{jz,d}$ denotes the aggregate use of capital good z at date t (the superscript d stands for demand).

Feasible Allocations and Perturbations. Here we define a feasible allocation.

. *(Feasible allocation). An allocation* $\left\{c_t^j, \iota_t^{zj}, y_t^j, k_t^{jz,d}, k_t^z\right\}$ *is feasible if equations* (2) *through* (5) *hold and the non-negativity constraints* $c_t^j \geq 0$, $\iota_t^{zj} \geq 0$, $y_t^j \geq 0$, $k_t^{jz,d} \geq 0$, *and* $k_t^z \geq 0$ *are satisfied.*

We assume throughout that preferences and technologies satisfy standard regularity conditions, so that all variables are smooth functions of a perturbation parameter θ. Derivatives such as $\frac{dc_t^j}{d\theta}$, $\frac{d\iota_t^{zj}}{d\theta}$, and $\frac{dk_t^{jz,d}}{d\theta}$ are therefore well defined.

Perturbations $d\theta$ may represent changes in production or accumulation technologies, in policies (e.g., taxes) or any other primitive of a fully specified model (e.g., borrowing constraints). Under this interpretation, the mapping from θ to allocations emerges endogenously and accounts for equilibrium effects. Alternatively, a perturbation may correspond to a direct change in feasible allocations selected by a planner — an interpretation useful to characterize efficient allocations, as in Section E.2.

Examples. The environment just described nests a wide range of settings involving accumulation, as the following five examples illustrate.

Example 1. [Neoclassical Growth Model]. The neoclassical growth model (Ramsey, 1928; Cass, 1965; Koopmans, 1965) is the workhorse model of capital accumulation. It features a deterministic economy with a single good ($J = 1$) and a single capital good ($Z = 1$), with preferences $V = \sum_t \beta^t u_t(c_t)$, after dropping indexes of unit dimensions. Production and accumulation technologies are $y_t = G_t\left(k_t^d\right)$ and $k_{t+1} = (1-\delta)\, k_t^d + \iota_t$

where δ denotes the depreciation rate. Resource constraints are $y_t = c_t + \iota_t$ and $k_t = k_t^d$.[4] Introducing adjustment costs (Uzawa, 1969; Hayashi, 1982) simply entails switching to $k_{t+1} = (1-\delta)\, k_t^d + \Upsilon_t\left(k_t^d, \iota_t\right)$, where $\Upsilon_t(\cdot)$ captures adjustment costs.

Example 2. [Investment-Specific Technological Change]. The one-sector Greenwood, Hercowitz, and Krusell (1997) model is the canonical environment to study investment-specific technological change, a leading instance of technological progress embodied in accumulation technologies. Relative to Example 1, this economy features two capital goods ($Z = 2$) instead of one, with a single good ($J = 1$) whose index we omit. The production technology becomes $y_t = G_t\left(k_t^{1,d}, k_t^{2,d}\right)$, and accumulation technologies are $k_{t+1}^1 = (1-\delta^1)\, k_t^{1,d} + \iota_t^1$ and $k_{t+1}^2 = (1-\delta^2)\, k_t^{2,d} + \theta\iota_t^2$, where θ scales investment in capital good 2 to capture an (unanticipated) investment-specific technology shock. Resource constraints are $y_t = c_t + \iota_t^1 + \iota_t^2$, $k_t^1 = k_t^{1,d}$, and $k_t^2 = k_t^{2,d}$.

Example 3. [Two-Sector Growth Model]. The two-sector growth model (Uzawa, 1965; Lucas, 1988; Rebelo, 1991) is the benchmark economy with multiple goods and capital goods. Relative to Example 2, this economy adds a second good ($J = 2$), with preferences $V = \sum_t \beta^t u_t\left(c_t^1\right)$ — good 2 serves only as an input into capital accumulation. Production technologies become $y_t^1 = G_t^1\left(k_t^{11,d}, k_t^{12,d}\right)$ and $y_t^2 = G_t^2\left(k_t^{21,d}, k_t^{22,d}\right)$, accumulation technologies are $k_{t+1}^1 = (1-\delta)\sum_j k_t^{j1,d} + \iota_t^1$ and $k_{t+1}^2 = (1-\delta)\sum_j k_t^{j2,d} + \iota_t^2$, and resource constraints are $y_t^1 = c_t^1 + \iota_t^1$, $y_t^2 = \iota_t^2$, $k_t^1 = k_t^{11,d} + k_t^{21,d}$, and $k_t^2 = k_t^{12,d} + k_t^{22,d}$.

Example 4. [Investment Network]. Horvath (2000) and vom Lehn and Winberry (2022) study a canonical investment network model. Relative to Example 3, this economy generalizes to an arbitrary number of goods and capital goods, $Z = J \geq 1$, with preferences $V = \sum_t \beta^t u_t\left(\left\{c_t^j\right\}_{j\in\mathcal{J}}\right)$. The production technology of good j is $y_t^j = G_t^j\left(k_t^{j,d}\right)$, and its accumulation technology is $k_{t+1}^j = (1-\delta^j)\, k_t^{j,d} + \mathfrak{I}_t^j\left(\left\{\iota_t^{j\ell}\right\}_{\ell\in\mathcal{J}}\right)$, where $\mathfrak{I}_t^j$ captures the investment network across goods used as investment inputs, with $\iota_t^j = \sum_\ell \iota_t^{\ell j}$. Resource constraints are $y_t^j = c_t^j + \iota_t^j$ and $k_t^j = k_t^{j,d}$.

Example 5. [Credit Cycles]. Kiyotaki and Moore (1997) introduce a benchmark economy in which a capital good in fixed supply is reallocated across technologies. Farming ($j = 1$) and gathering ($j = 2$) produce two perfectly substitutable goods ($J = 2$)

[4] The neoclassical growth model often features capital reversibility. Section E.4 of the Online Appendix shows that capital reversibility can be represented using two production technologies.

using a single capital good ($Z = 1$), land, whose index we omit.[5] Production technologies are $y_t^1 = G_t^1\left(k_t^{1,d}\right)$ and $y_t^2 = G_t^2\left(k_t^{2,d}\right)$, where $G_t^1(\cdot)$ is linear (farming) and $G_t^2(\cdot)$ is strictly concave (gathering). Because land neither depreciates nor can be produced, the accumulation technology is $k_{t+1} = k_t^{1,d} + k_t^{2,d} = \bar{k}$ at all dates. Hence, $\iota_t^j = 0$ for all j, and resource constraints reduce to $y_t^1 + y_t^2 = c_t^1 + c_t^2$ and $k_t = k_t^{1,d} + k_t^{2,d}$.[6]

We conclude the description of the environment by highlighting its generality beyond the examples presented above.

Remark 1. *(Generality of the environment)* The environment is sufficiently general to nest a broad class of settings involving accumulation. In particular, it accommodates (i) accumulation technologies that map capital and investment across multiple dates, capturing time-to-build (Kydland and Prescott, 1982); (ii) capital goods valued directly by individuals, which can be modeled through technologies that transform capital into goods; (iii) interactions across distinct capital goods, so that excluding other capital goods from a given accumulation equation is without loss of generality, since such interactions can be represented through production technologies that generate investment for other capital types; (iv) variable capital utilization, through a suitable specification of (3); and (v) human capital, understood as a particular type of capital good.

3 Origins of Welfare Gains

We now show that the welfare gains from any perturbation in the environment considered here are solely driven by production efficiency gains, which can be attributed to changes in allocations (allocative efficiency), or technological changes in production or accumulation (technical efficiency). The allocative gains can in turn be due to: (i) reallocating investment towards different capital goods (cross-sectional investment efficiency); (ii) adjusting the aggregate level of investment (aggregate investment efficiency); (iii)

[5] Kiyotaki and Moore (1997) feature a single homogeneous good (fruit). We equivalently represent it with two perfectly substitutable goods, one produced by each technology. This model also features two types of individuals ($I = 2$) — impatient farmers and patient gatherers — but individual heterogeneity does not impact the aggregates described here.

[6] Cross-sectional capital use efficiency is the only active allocative component in models in which capital is in fixed supply and its productivity depends on who uses it. Other examples include Section 2 of Eisfeldt and Rampini (2008), the baseline model of Fuchs, Green, and Papanikolaou (2016), and Dávila, Jeenas, and Philippon (2026).

reallocating capital towards the production of different goods (cross-sectional capital use efficiency).

3.1 Welfare Assessment

The welfare assessment of any perturbation with a representative individual is given by the change in utility $\frac{dV}{d\theta}$. Dividing by a normalizing factor λ, with units $[\lambda] = \frac{\text{utils}}{\text{welfare numeraire}}$, expresses the assessment in units of a chosen welfare numeraire, $\frac{dV^{\lambda}}{d\theta} = \frac{dV}{d\theta}/\lambda$.[7] To compare welfare gains across dates, we similarly normalize date-t utility gains by a factor λ_t, with units $[\lambda_t] = \frac{\text{utils}}{\text{date-}t\text{ welfare numeraire}}$, and write

$$\frac{dV^{\lambda}}{d\theta} = \frac{\frac{dV}{d\theta}}{\lambda} = \sum_t \omega_t \frac{dV_t^{\lambda}}{d\theta}, \quad \text{where} \quad \omega_t = \frac{\lambda_t}{\lambda} \tag{6}$$

is the marginal rate of substitution between the date-t and the lifetime welfare numeraires (a time discount factor). The date-t flow gain

$$\frac{dV_t^{\lambda}}{d\theta} = \sum_j AMRS_{t,c}^j \frac{dc_t^j}{d\theta}, \quad \text{where} \quad AMRS_{t,c}^j = \frac{\beta^t \frac{\partial u_t}{\partial c_t^j}}{\lambda_t}, \tag{7}$$

values changes in consumption at the aggregate marginal rate of substitution between good j and the date-t numeraire, $AMRS_{t,c}^j$.[8] Since welfare is determined entirely by changes in final aggregate consumption, Lemma 1 reformulates these results in terms of production efficiency.

Lemma 1. *(Welfare as Production Efficiency) In deterministic economies with a single*

[7]Common choices of welfare numeraires are aggregate perpetual consumption (Lucas, 1987; Alvarez and Jermann, 2004), date-0 aggregate consumption, and date-0 consumption. These choices respectively imply

$$\lambda = \sum_t \beta^t \sum_j \frac{\partial u_t}{\partial c_t^j} c_t^j, \quad \lambda = \sum_j \frac{\partial u_0}{\partial c_0^j} c_0^j, \quad \text{and} \quad \lambda = \sum_j \frac{\partial u_0}{\partial c_0^j}.$$

A nominal numeraire can be accommodated as well, as in Section 4. Section E.3 of the Online Appendix shows how to derive (6) from a global consumption-equivalent formulation and how to compute λ for different welfare numeraires. Section B of the Online Appendix shows how to make welfare assessments for welfarist planners with heterogeneous individuals.

[8]We add the qualifier aggregate to the $AMRS$ to highlight that it is a marginal rate of substitution for aggregate consumption. This distinction becomes relevant in models with heterogeneous individuals, as explained in Section B of the Online Appendix.

individual, welfare assessments are solely driven by production efficiency, given by

$$\underbrace{\frac{dV^\lambda}{d\theta}}_{\substack{Welfare\\ Gains}} = \underbrace{\Xi^{E,P}}_{\substack{Production\\ Efficiency}} = \sum_t \omega_t \underbrace{\sum_j AMRS^j_{t,c} \frac{dc^j_t}{d\theta}}_{=\Xi^{E,P}_t} = \sum_t \omega_t \Xi^{E,P}_t, \tag{8}$$

where ω_t and $AMRS^j_{t,c}$ are defined in (6) *and* (7).

Equation (8) simply states that welfare gains can be expressed as the discounted value — using ω_t — of the date-t changes in aggregate consumption of each of the goods — each contemporaneously valued at $AMRS^j_{t,c}$. Therefore, identifying the origins of welfare/production efficiency gains boils down to tracing back changes in aggregate consumption c^j_t to primitive changes in technology, and capital and investment allocations.

Shares Definitions. Working with shares, rather than levels, is critical to distinguish gains due to reallocation from those due to changes in aggregates. Formally, changes in capital and investment uses can be expressed as

$$\frac{dk^{jz,d}_t}{d\theta} = \frac{d\chi^{jz,d}_{t,k}}{d\theta} k^{z,d}_t + \chi^{jz,d}_{t,k} \frac{dk^{z,d}_t}{d\theta} \quad \text{and} \quad \frac{d\iota^{zj}_t}{d\theta} = \frac{d\chi^{zj}_{t,\iota}}{d\theta} \iota^j_t + \chi^{zj}_{t,\iota} \frac{d\iota^j_t}{d\theta}, \tag{9}$$

where the date-t *capital use share* of capital good z used to produce good j is given by $\chi^{jz,d}_{t,k} = k^{jz,d}_t / k^{z,d}_t$, and the date-$t$ *investment use share* of good j used to invest in capital good z is given by $\chi^{zj}_{t,\iota} = \iota^{zj}_t / \iota^j_t$.

We also define good j's date-t *investment share* by $\phi^j_{t,\iota} = \iota^j_t / y^j_t$, which represents the share of good j's date-t aggregate supply y^j_t devoted to investment. Therefore, changes in aggregate investment of good j can be expressed as

$$\frac{d\iota^j_t}{d\theta} = \frac{d\phi^j_{t,\iota}}{d\theta} y^j_t + \phi^j_{t,\iota} \frac{dy^j_t}{d\theta}. \tag{10}$$

3.2 Propagation and Welfare-Relevant Statistics

To understand the underlying forces that determine production efficiency, it is necessary to first understand how perturbations impact the allocations of capital and goods. We do so by introducing a propagation matrix for capital — the capital inverse matrix — and two propagation matrices for goods — the goods and goods-capital inverse matrices. These matrices characterize how perturbations propagate through capital and goods allocations over time. Relying on these propagation matrices, we define welfare-relevant statistics that translate changes in allocations and technologies into welfare changes.

3.2.1 Capital Propagation: Capital Inverse Matrix

Lemma 2 derives the capital inverse matrix $\boldsymbol{\mathcal{A}}_k$, which characterizes the ultimate change in the aggregate supply of capital goods induced by unit impulses in the supply of capital goods at all dates.[9]

Lemma 2. *(Capital Inverse Matrix) Changes in the aggregate supply of capital good z, $\frac{dk^z_{t+1}}{d\theta}$ ($\frac{d\boldsymbol{k}}{d\theta}$), can be expressed in terms of changes in accumulation technology $\frac{\partial H^z_{t+1}}{\partial\theta}$ ($\boldsymbol{H}_\theta$), changes in capital use shares $\frac{d\chi^{jz,d}_{t,k}}{d\theta}$ ($\frac{d\boldsymbol{\chi}_{k^d}}{d\theta}$), and changes in investment use $\frac{d\iota^{zj}_t}{d\theta}$ ($\frac{d\mathring{\boldsymbol{\iota}}}{d\theta}$), as*

$$\frac{d\boldsymbol{k}}{d\theta} = \underbrace{\boldsymbol{\mathcal{A}}_k}_{\substack{\text{Accumulation}\\ \text{Propagation}}} \underbrace{\left(\boldsymbol{H}_k \frac{d\boldsymbol{\chi}_{k^d}}{d\theta}\boldsymbol{k}^d + \boldsymbol{H}_\iota \frac{d\mathring{\boldsymbol{\iota}}}{d\theta} + \boldsymbol{H}_\theta\right)}_{\substack{\text{Accumulation}\\ \text{Impulse}}}, \tag{11}$$

where $\frac{d\boldsymbol{k}}{d\theta}$ denotes the $Z(T+1)\times 1$ vector of $\frac{dk^z_t}{d\theta}$, and

$$(\textit{Capital Inverse Matrix}) \qquad \boldsymbol{\mathcal{A}}_k = \left(\boldsymbol{I} - \boldsymbol{H}_k\boldsymbol{\chi}_{k^d}\right)^{-1},$$

defines the $Z(T+1)\times Z(T+1)$ capital inverse matrix. The remaining matrices are defined in Appendix A.

Lemma 2 characterizes how the aggregate supply of capital goods ultimately changes in response to the three "accumulation impulse" terms, which represent the first-round impact of the perturbation on the supply of capital goods. A perturbation that changes capital use shares at date $t-1$ by $\frac{d\chi^{jz,d}_{t-1,k}}{d\theta}$ mechanically increases the amount of capital use $k^{jz,d}_{t-1}$ in proportion to $k^{z,d}_{t-1}$, which in turn increases the supply of capital good z at date t by $\frac{\partial H^z_t}{\partial k^{jz,d}_{t-1}}$. Similarly, a perturbation that changes investment use at date $t-1$ by $\frac{d\iota^{zj}_{t-1}}{d\theta}$ mechanically increases the supply of capital good z at date t by $\frac{\partial H^z_t}{\partial \iota^{zj}_{t-1}}$. Changes in the accumulation technology of capital good z at date t mechanically increase the aggregate supply of capital good z at date t by $\frac{\partial H^z_t}{\partial\theta}$.

[9] To simplify the exposition, we denote all identity matrices by $\boldsymbol{I}$; their dimensions are conformable to the relevant expression and are suppressed throughout, except in the Appendix.

Such first-round changes in the aggregate supply of capital goods at date t induce further changes in the aggregate use of capital goods at date t, which in turn induce further changes in the aggregate supply of capital at date $t+1$. These dynamic knock-on effects summarize the accumulation propagation mechanism and are captured by the capital inverse matrix $\boldsymbol{\mathcal{A}}_k$.[10]

Example. (Neoclassical Growth Model). In the neoclassical growth model, the accumulation technology is $k_{t+1} = (1-\delta)\,k_t^d + \iota_t$, so $\frac{\partial H_{t+1}}{\partial k_t^d} = 1-\delta$, and capital has a single use, so $\chi_{t,k} = 1$. The matrix $\boldsymbol{H}_k \boldsymbol{\chi}_{k^d}$ is then a one-period-lag operator with entries $1-\delta$, so the capital inverse matrix is

$$\boldsymbol{\mathcal{A}}_k = \left(\boldsymbol{I} - \boldsymbol{H}_k \boldsymbol{\chi}_{k^d}\right)^{-1} = \begin{pmatrix} 1 & & & & \\ 1-\delta & 1 & & & \\ (1-\delta)^2 & 1-\delta & 1 & & \\ \vdots & & \ddots & \ddots & \\ (1-\delta)^T & (1-\delta)^{T-1} & \cdots & 1-\delta & 1 \end{pmatrix}.$$

The entry in row s and column t, given by $(1-\delta)^{s-t}$ for $s \geq t$, measures the increase in the supply of capital at date s induced by a unit impulse to the supply of capital at date t: the fraction of the impulse that survives depreciation between dates t and s. With full depreciation ($\delta = 1$), $\boldsymbol{\mathcal{A}}_k = \boldsymbol{I}$, so impulses to the supply of capital do not propagate over time. Without depreciation ($\delta = 0$), all entries on and below the diagonal equal one, so impulses to the supply of capital are permanent.

3.2.2 Goods Propagation: Goods and Goods-Capital Inverse Matrices

Lemma 3 derives two propagation matrices — the goods and goods-capital inverse matrices — that characterize how changes in technology and input allocations propagate and accumulate through the economy to determine the aggregate supply of goods.

Lemma 3. *(Goods Inverse and Goods-Capital Inverse Matrices) Changes in the aggregate supply of good j, $\frac{dy_t^j}{d\theta}$ ($\frac{d\boldsymbol{y}}{d\theta}$), can be expressed in terms of changes in production technology $\frac{\partial G_t^j}{\partial \theta}$ ($\boldsymbol{G}_\theta$), changes in capital use shares $\frac{d\chi_{t,k}^{jz,d}}{d\theta}$ ($\frac{d\boldsymbol{\chi}_{k^d}}{d\theta}$), changes in accumulation technology*

[10] The capital inverse matrix $\boldsymbol{\mathcal{A}}_k$ is nonnegative and block lower triangular with identity diagonal blocks: capital propagates only forward in time, while contemporaneous capital supply shocks pass one for one.

$\frac{\partial H^z_{t+1}}{\partial\theta}$ ($\boldsymbol{H}_\theta$), changes in investment use shares $\frac{d\chi^{zj}_{t,\iota}}{d\theta}$ ($\frac{d\boldsymbol{\chi}_\iota}{d\theta}$), and changes in investment shares $\frac{d\phi^j_{t,\iota}}{d\theta}$ ($\frac{d\boldsymbol{\phi}_\iota}{d\theta}$), as

$$\frac{d\boldsymbol{y}}{d\theta} = \underbrace{\boldsymbol{\Psi}_y}_{\substack{\text{Goods}\\ \text{Propagation}}} \underbrace{\left(\boldsymbol{G}_k \frac{d\boldsymbol{\chi}_{k^d}}{d\theta}\boldsymbol{k}^d + \boldsymbol{G}_\theta\right)}_{\substack{\text{Goods}\\ \text{Impulse}}} + \underbrace{\boldsymbol{\Psi}_k}_{\substack{\text{Goods-Capital}\\ \text{Propagation}}} \underbrace{\left(\boldsymbol{H}_k \frac{d\boldsymbol{\chi}_{k^d}}{d\theta}\boldsymbol{k}^d + \boldsymbol{H}_\iota \frac{d\boldsymbol{\chi}_\iota}{d\theta}\boldsymbol{\iota} + \boldsymbol{H}_\iota\boldsymbol{\chi}_\iota \frac{d\boldsymbol{\phi}_\iota}{d\theta}\boldsymbol{y} + \boldsymbol{H}_\theta\right)}_{\substack{\text{Goods-Capital}\\ \text{Impulse}}}, \tag{12}$$

where $\frac{d\boldsymbol{y}}{d\theta}$ denotes the $J(T+1)\times 1$ vector of $\frac{dy^j_t}{d\theta}$,

$$(\textit{Goods Inverse Matrix}) \qquad \boldsymbol{\Psi}_y = \left(\boldsymbol{I} - \boldsymbol{G}_k\boldsymbol{\chi}_{k^d}\boldsymbol{\mathcal{A}}_k\boldsymbol{H}_\iota\boldsymbol{\chi}_\iota\boldsymbol{\phi}_\iota\right)^{-1}, \tag{13}$$

defines the $J(T+1)\times J(T+1)$ goods inverse matrix, and

$$(\textit{Goods-Capital Inverse Matrix}) \qquad \boldsymbol{\Psi}_k = \boldsymbol{\Psi}_y\boldsymbol{G}_k\boldsymbol{\chi}_{k^d}\boldsymbol{\mathcal{A}}_k, \tag{14}$$

defines the $J(T+1)\times Z(T+1)$ goods-capital inverse matrix. The remaining matrices are defined in Appendix A.

Lemma 3 characterizes how the aggregate supply of goods ultimately changes in response to changes in goods or capital. The first term in the right-hand side of (12) characterizes how the aggregate supply of goods ultimately changes in response to two "goods impulse" terms, which represent the first-round impact of the perturbation on the supply of goods. A perturbation that changes capital use shares at date t by $\frac{d\chi^{jz,d}_{t,k}}{d\theta}$ mechanically increases the amount of capital use $k^{jz,d}_t$ in proportion to $k^{z,d}_t$, which in turn increases the supply of good j at date t by $\frac{\partial G^j_t}{\partial k^{jz,d}_t}$. Changes in the production technology of good j at date t mechanically increase the aggregate supply of good j at date t by $\frac{\partial G^j_t}{\partial\theta}$.

The first-round changes in the supply of goods at date t mechanically change aggregate investment at date t (for a given investment share), which induce further changes in the aggregate supply of capital goods at date $t+1$, which in turn affect the aggregate supply of goods at date $t+1$, and so on. These knock-on effects summarize the dynamic linkages between goods and are captured by the goods inverse matrix $\boldsymbol{\Psi}_y$.[11]

The second term in the right-hand side of (12) characterizes how the aggregate supply of goods ultimately changes in response to the four "goods-capital impulse" terms, which

[11]The goods inverse matrix generalizes the one introduced in Dávila and Schaab (2023) to economies with capital accumulation and investment.

represent the first-round impact of the perturbation on the supply of capital goods. As discussed previously, changes in the accumulation technology and in capital use shares in proportion to aggregate capital use directly change the supply of capital goods. Furthermore, changes in the investment use shares at date $t-1$ mechanically increase the amount of investment use ι_{t-1}^{zj} in proportion to ι_{t-1}^{j}, which directly increases the supply of capital good z at date t by $\frac{\partial H_t^z}{\partial \iota_{t-1}^{zj}}$, and changes in the investment shares at date $t-1$ mechanically increase the amount of aggregate investment ι_{t-1}^{j} in proportion to y_{t-1}^{j}, which directly increases the supply of capital good z at date t by $\sum_j \chi_{t-1,\iota}^{zj} \frac{\partial H_t^z}{\partial \iota_{t-1}^{zj}}$. Equation (12) then shows that such first-round effects on the supply of capital goods directly raise the supply of good j at date t by $\sum_z \chi_{t,k}^{jz,d} \frac{\partial G_t^j}{\partial k_t^{jz,d}}$.

As previously discussed, $\boldsymbol{\mathcal{A}}_k$ characterizes the ultimate change in the aggregate supply of capital goods induced by unit impulses in the supply of capital goods at all dates, where the latter mechanically raises the aggregate supply of goods by $\boldsymbol{G}_k \boldsymbol{\chi}_{k^d}$. Hence, the first-round changes in the supply of goods induced by the ultimate changes in the aggregate supply of capital goods is captured by $\boldsymbol{G}_k \boldsymbol{\chi}_{k^d} \boldsymbol{\mathcal{A}}_k$, and the corresponding ultimate effects on the aggregate supply of goods are captured by $\boldsymbol{\Psi}_y \boldsymbol{G}_k \boldsymbol{\chi}_{k^d} \boldsymbol{\mathcal{A}}_k$, which is exactly the goods-capital inverse matrix $\boldsymbol{\Psi}_k$.[12]

In a nutshell, the goods inverse matrix, $\boldsymbol{\Psi}_y$, characterizes the ultimate change in the aggregate supply of goods induced by unit impulses in the supply of goods, whereas the goods-capital inverse matrix, $\boldsymbol{\Psi}_k$, characterizes the ultimate change in the aggregate supply of goods induced by unit impulses in the supply of capital goods. The following remark highlights how the goods inverse matrix differs from propagation matrices identified in the literature.

Remark 2. *(Inverse Matrices $\boldsymbol{\mathcal{A}}_k$, $\boldsymbol{\Psi}_y$, and $\boldsymbol{\Psi}_k$ are Purely Technological)* While propagation matrices abound in the study of models with rich production structures, the goods and goods-capital inverse matrices introduced in Lemma 3 are purely technological objects at a given allocation.[13] That is, $\boldsymbol{\Psi}_y$ and $\boldsymbol{\Psi}_k$, as well as the capital inverse matrix $\boldsymbol{\mathcal{A}}_k$, are exclusively based on production and accumulation technologies, and are defined

[12]Unlike the capital inverse matrix, $\boldsymbol{\Psi}_y$ and $\boldsymbol{\Psi}_k$ typically do not admit closed-form expressions. Nonetheless, for the same reason as the capital inverse matrix, both are block lower triangular: every propagation channel runs forward in time. The diagonal blocks of $\boldsymbol{\Psi}_y$ are identity matrices, while those of $\boldsymbol{\Psi}_k$ equal the corresponding blocks of $\boldsymbol{G}_k \boldsymbol{\chi}_{k^d}$, reflecting that capital supplied at date-t raises the supply of goods contemporaneously.

[13]The allocations at which the goods inverse matrix is evaluated can themselves be endogenously determined within a particular economic or market structure.

without relying on prices or competitive notions.

3.2.3 Welfare-Relevant Statistics

Finally, understanding the origins of production efficiency gains requires defining the welfare-relevant statistics that translate changes in allocations and technologies into welfare changes: MSV, MWP, and $AMWP$.

1. *(Marginal Social Value) The marginal social values of good j and capital good z available at date t, $MSV_{t,y}^{j}$ and $MSV_{t,k}^{z}$, are defined as the elements of the $1 \times J(T+1)$ and $1 \times Z(T+1)$ vectors $\boldsymbol{MSV}_y$ and $\boldsymbol{MSV}_k$, given by*

$$\boldsymbol{MSV}_y = \boldsymbol{\omega} \boldsymbol{AMRS}_c \boldsymbol{\phi}_c \boldsymbol{\Psi}_y \quad \text{and} \quad \boldsymbol{MSV}_k = \boldsymbol{\omega} \boldsymbol{AMRS}_c \boldsymbol{\phi}_c \boldsymbol{\Psi}_k, \tag{15}$$

where $\boldsymbol{\omega}$ is the $1 \times (T+1)$ vector of ω_t, $\boldsymbol{AMRS}_c$ is the $(T+1) \times J(T+1)$ matrix of $AMRS_{t,c}^{j}$ for all goods at all dates, $\boldsymbol{\phi}_c$ is the $J(T+1) \times J(T+1)$ matrix of the consumption shares $\phi_{t,c}^{j} = c_t^j / y_t^j$ for all goods at all dates, $\boldsymbol{\Psi}_y$ is the $J(T+1) \times J(T+1)$ goods inverse matrix, and $\boldsymbol{\Psi}_k$ is the $J(T+1) \times Z(T+1)$ goods-capital inverse matrix.

The marginal social value of good j (capital good z) available at date t captures the welfare gain associated with having an additional unit of that good (capital good) leaving shares unchanged. While a unit impulse in the supply of goods (capital goods) generates an ultimate increase in the aggregate supply of goods given by the goods inverse matrix $\boldsymbol{\Psi}_y$ (goods-capital inverse matrix $\boldsymbol{\Psi}_k$), only the aggregate consumption share $\boldsymbol{\phi}_c$ is consumed by the individual. $\boldsymbol{AMRS}_c$ captures the gain associated with increasing aggregate consumption expressed in units of the date-t welfare numeraire. This definition highlights that the social value of a good or a capital good emanates from the final consumption — potentially of other goods at different dates — it ultimately generates.

2. *(Marginal Welfare Product) The marginal welfare products of investment z and capital z for technology j at date t are given by*

$$MWP_{t,\iota}^{zj} = MSV_{t+1,k}^{z} \frac{\partial H_{t+1}^{z}}{\partial \iota_t^{zj}} \quad \text{and} \quad MWP_{t,k}^{jz} = MSV_{t,y}^{j} \frac{\partial G_t^j}{\partial k_t^{jz,d}} + MSV_{t+1,k}^{z} \frac{\partial H_{t+1}^{z}}{\partial k_t^{jz,d}}. \tag{16}$$

Marginal welfare products capture the welfare gains associated with increasing investment or using a capital good. Marginal increases in ι_t^{zj} increase the amount of capital good z

available at date $t+1$ at impact by the corresponding technological marginal accumulation product, $\frac{\partial H^z_{t+1}}{\partial \iota^{zj}_t}$. As just described, the social value of a capital good at date $t+1$ is summarized by $MSV^z_{t+1,k}$. Hence, the marginal welfare product of investment is given by the product of the marginal accumulation product of investment and the marginal social value of capital.

Marginal increases in $k^{jz,d}_t$ have two effects. First, they increase the supply of good j at date t at impact by the technological marginal product of capital, $\frac{\partial G^j_t}{\partial k^{jz,d}_t}$. Second, they increase the amount of capital good z available at date $t+1$ at impact by the corresponding technological marginal accumulation product, $\frac{\partial H^z_{t+1}}{\partial k^{jz,d}_t}$. The social value of the first effect is equal to the product of the social value of good j available at date t, given by $MSV^j_{t,y}$, and the marginal product of capital. The social value of the second effect is equal to the product of the social value of capital good z available at date $t+1$, given by $MSV^z_{t+1,k}$, and the marginal accumulation product of capital. And the marginal welfare product of capital is the sum of the social value of both effects.

3. *(Aggregate Marginal Welfare Product) The aggregate marginal welfare products of investment j and capital z at date t are given by*

$$AMWP^j_{t,\iota} = \sum_z \chi^{zj}_{t,\iota} MWP^{zj}_{t,\iota} \quad \textit{and} \quad AMWP^z_{t,k} = \sum_j \chi^{jz,d}_{t,k} MWP^{jz}_{t,k}. \tag{17}$$

The aggregate marginal welfare product of investment for technology j at date t corresponds to the efficiency gain associated with increasing the amount of aggregate investment of good j in proportion to the investment use shares. The aggregate marginal welfare product of capital for capital good z at date t corresponds to the efficiency gain associated with increasing the use of capital z in proportion to the capital use shares.

3.3 Origins of Production Efficiency Gains

Once we have defined the welfare-relevant statistics, we are ready to present the first main result of the paper. Theorem 1 shows that it is possible to attribute all production efficiency gains to allocative efficiency gains due to adjusting investment and capital uses as well as technical efficiency gains from primitive changes in technologies.

Theorem 1. *(Production Efficiency Decomposition) Production efficiency gains, $\Xi^{E,P}$, can be attributed to (i) cross-sectional investment efficiency, (ii) aggregate investment* *efficiency, (iii) cross-sectional capital use efficiency, (iv) accumulation technology change, and (v) production technology change, as follows*

$$\Xi^{E,P} = \overbrace{\underbrace{\sum_t \sum_j \mathbb{C}ov_z^{\Sigma}\left[MWP_{t,\iota}^{zj}, \frac{d\chi_{t,\iota}^{zj}}{d\theta}\right]\iota_t^j}_{\substack{\text{Cross-Sectional}\\ \text{Investment Efficiency}}} + \underbrace{\sum_t \sum_j \left(AMWP_{t,\iota}^j - \omega_t AMRS_{t,c}^j\right)\frac{d\phi_{t,\iota}^j}{d\theta} y_t^j}_{\substack{\text{Aggregate}\\ \text{Investment Efficiency}}}}^{\text{Investment Efficiency}}$$
$$+ \overbrace{\underbrace{\sum_t \sum_z \mathbb{C}ov_j^{\Sigma}\left[MWP_{t,k}^{jz}, \frac{d\chi_{t,k}^{jz,d}}{d\theta}\right]k_t^{z,d}}_{\substack{\text{Cross-Sectional}\\ \text{Capital Use Efficiency}}}}^{\text{Capital Efficiency}} + \underbrace{\sum_t \sum_z MSV_{t,k}^z \frac{\partial H_t^z}{\partial \theta}}_{\substack{\text{Accumulation}\\ \text{Technology Change}}} + \underbrace{\sum_t \sum_j MSV_{t,y}^j \frac{\partial G_t^j}{\partial \theta}}_{\substack{\text{Production}\\ \text{Technology Change}}}, \tag{18}$$

where $\mathbb{C}ov_z^{\Sigma}[\cdot,\cdot] = Z \cdot \mathbb{C}ov_z[\cdot,\cdot]$ and $\mathbb{C}ov_j^{\Sigma}[\cdot,\cdot] = J \cdot \mathbb{C}ov_j[\cdot,\cdot]$ denote cross-sectional covariance-sums among capital and goods, respectively.

Cross-sectional components correspond to covariances across uses, measuring gains from reallocating investment or capital goods from low to high marginal welfare product uses, for given levels of aggregate investment or capital use. The aggregate investment efficiency component measures the gains associated with adjusting the share of the aggregate supply of goods devoted to investment, for a given level of aggregate supply. Consequently, for good j available at date t, it is determined by the product of the difference of the value of investing relative to consuming, $AMWP_{t,\iota}^j - \omega_t AMRS_{t,c}^j$, and the change in the investment share, $\frac{d\phi_{t,\iota}^j}{d\theta} y_t^j$. These three components capture allocative efficiency gains since they are associated with the reallocation of resources across the economy.

The final two components are associated with efficiency gains due to primitive changes in technologies for given allocations. The gain from changes in the production technology of good j available at date t is given by its marginal social value, $MSV_{t,y}^j$. The gain from changes in the accumulation technology of capital good z available at date t is given by its marginal social value, $MSV_{t,k}^z$. These last two components capture technical efficiency gains since they are associated with changes in technologies.

Corollary 1 presents several properties that production efficiency satisfies. These are helpful to understand the economics behind each term, and to justify each of the labels.

Corollary 1. *(Properties of Production Efficiency Decomposition)*

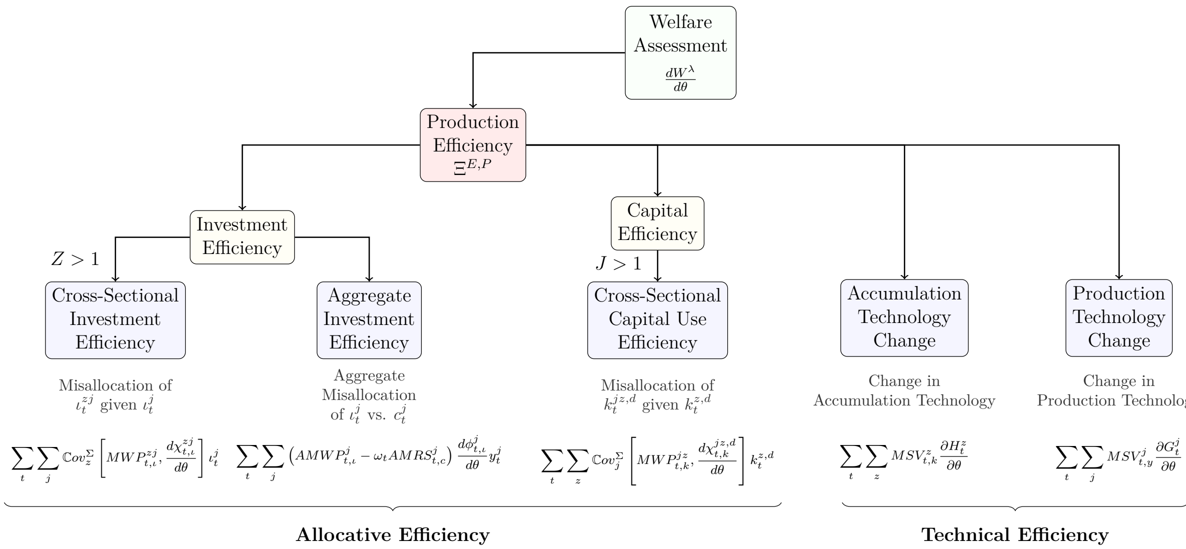


Figure 1: Production Efficiency Decomposition

Note: This diagram illustrates the production efficiency decomposition introduced in this paper. Theorem 1 decomposes production efficiency in deterministic economies with a single individual into investment efficiency, capital efficiency, accumulation technology change, and production technology change. Theorem 3 in Section B generalizes this decomposition to economies with intermediate inputs and factors; the corresponding diagram is reported in Figure OA1.

(a) *(Single Good Economies) In economies with a single good ($J=1$), cross-sectional capital use efficiency is zero.*

(b) *(Single Capital Good Economies) In economies with a single capital good ($Z=1$), cross-sectional investment efficiency is zero.*

(c) *(Specialized Investment/Capital Economies) In economies in which all capital goods (investment) are specialized with $\chi_{t,k}^{jz,d}=1$ $\left(\chi_{t,\iota}^{zj}=1\right)$ for all z (j) at all dates, cross-sectional capital use (investment) efficiency is zero.*

(d) *(Equalized $MWP_{t,\iota}^{zj}$ or $MWP_{t,k}^{jz}$) If marginal welfare products of capital (investment) are equalized across all goods j (capital goods z) with $k_t^d>0$ $\left(\iota_t^j>0\right)$ at all dates, cross-sectional capital use (investment) efficiency is zero.*

(e) *(Fixed Investment Shares Economies) In economies in which all investment shares, $\phi_{t,\iota}^j$, are fixed for all j at all dates, aggregate investment efficiency is zero.*

In Section E.7 of the Online Appendix, we apply Theorem 1 to each of the five examples described in Section 2. Table 1 summarizes which components are active in each of these examples.

Table 1: Allocative Production Efficiency

Example	Cross-Sectional Investment Efficiency	Aggregate Investment Efficiency	Cross-Sectional Capital Use Efficiency
Neoclassical Growth/ Adjustment Costs	$= 0$	✓	$= 0$
Investment-Specific Technological Change	✓	✓	$= 0$
Two-Sector Growth Model	$= 0$	✓	✓
Investment Network	✓	✓	$= 0$
Credit Cycles	$= 0$	$= 0$	✓

Note: This table illustrates which components of the production efficiency decomposition in Theorem 1 are non-zero in each of the examples. Both technology change components — accumulation and production — are possible in all examples, so the table only reports the allocative components.

Insights from Production Efficiency Decomposition. The following remarks highlight insights that emerge from the production efficiency decomposition.

Remark 3. *(Technological and Preference Origins of Production Efficiency Gains).* Theorem 1 traces the origins of efficiency gains under any perturbation to changes in the allocation of resources and to primitive changes in technology. Since this decomposition is purely based on preferences, technologies, and resource constraints, it is useful to quantify, compare, and contrast different economic environments, e.g., competitive, strategic, search, bargaining, or contracting. Note also that none of our results so far — Lemmas 2 and 3 and Theorem 1 — rely on any optimality conditions (i.e., the envelope theorem).

Remark 4. *(Social Value of Changes in Accumulation or Production Technologies).* Theorem 1 shows that the efficiency gains from pure technological change are summarized by $MSV^{j}_{t,y}$ and $MSV^{z}_{t,k}$, without making assumptions about individual behavior, budget constraints, prices, or equilibrium notions. The technology change components are always positive if technology improves since $MSV^{j}_{t,y} > 0$ and $MSV^{z}_{t,k} > 0$. However, production efficiency can be negative after a technological improvement if its impact on allocative efficiency is sufficiently negative, which can only happen at inefficient allocations.

Remark 5. *(Rationale and Uniqueness of the Decomposition).* By design, the allocative efficiency components are systematically expressed in terms of changes in allocation shares. Working with shares allows us to separate changes due to reallocation (holding aggregate investment, goods supply, and capital use fixed) from changes in aggregates. This separation would not be possible if the decomposition were expressed in levels. This is the unique decomposition that, at each step, separates changes in variables into components reflecting changes in shares and changes in aggregates.

Remark 6. *(Efficiency Conditions).* The decomposition in Theorem 1 can also be used to characterize the set of Pareto efficient allocations. As we show in Appendix E.2, an allocation is efficient when no feasible perturbation of investment, investment use, or capital use shares makes any allocative efficiency component positive. Theorem 4 translates this requirement into conditions on marginal welfare products, providing, to our knowledge, the first characterization of efficiency conditions in economies with capital and investment at this level of generality.

4 Frictionless Competition

Up to this point, the analysis has remained agnostic about optimizing behavior and equilibrium. We now turn to frictionless competitive economies, where we exploit optimality conditions to show that equilibrium prices encode the welfare-relevant statistics identified in Theorem 1. This allows us to define a new set of Domar weights that determine the pure impact of changes to accumulation technologies — *accumulation* Domar weights — in addition to the well-understood production Domar weights. A central result of this section is Corollary 3, which shows that the accumulation Domar sum is larger than the production Domar sum.

4.1 Competitive Equilibrium

We consider a competitive equilibrium in which the individual consumes goods, owns and accumulates capital, and rents capital to firms who operate production technologies. Formally, the individual utility maximization problem is given by

$$V = \max_{\left\{c_t^j, \iota_t^{zj}, k_t^{jz,s}\right\}} \sum_t \beta^t u_t \left(\left\{ c_t^j \right\}_{j \in \mathcal{J}} \right),$$

subject to the predetermined date-0 supply of capital goods k_0^z, accumulation technologies (3), and the date-t budget constraint

$$\sum_j p_t^j \left(c_t^j + \sum_z \iota_t^{zj} \right) = \sum_z r_t^z k_t^z + \sum_j \pi_t^j,$$

where p_t^j denotes the price of good j at date t, r_t^z denotes the rental rate of capital good z at date t, and π_t^j denotes the profits of production technology j at date t.

Firms operating production technology j rent capital to maximize their date-t profits, given by

$$\pi_t^j = \max_{\left\{k_t^{jz,d}\right\}} p_t^j y_t^j - \sum_z r_t^z k_t^{jz,d}.$$

Since firms solve a sequence of static problems, all intertemporal decisions rest with the individual, who owns and accumulates the capital stock.

4. *(Frictionless Competitive Equilibrium) Given the date-*0 *supply of capital goods* $\{k_0^z\}_{z\in\mathcal{Z}}$ *and taking the nominal numeraire to be the date-t welfare numeraire, a frictionless competitive equilibrium is given by a feasible allocation* $\left\{c_t^j, \iota_t^{zj}, y_t^j, k_t^{jz,d}, k_t^z\right\}$ *and prices* $\left\{p_t^j, q_t^z, r_t^z\right\}$ *that satisfy resource constraints for goods and capital* (4) *and* (5)*, such that the following conditions hold:*

(i) *The aggregate marginal rates of substitution satisfy:* $AMRS_{t,c}^j = p_t^j$.

(ii) *Investment satisfies the optimality condition:* $p_t^j = \frac{\omega_{t+1}}{\omega_t} q_{t+1}^z \frac{\partial H_{t+1}^z}{\partial \iota_t^{zj}}$.

(iii) *The capital Euler equations hold:* $q_t^z = r_t^z + \frac{\omega_{t+1}}{\omega_t} q_{t+1}^z \frac{\partial H_{t+1}^z}{\partial k_t^{jz,s}}$.

(iv) *Firms operate technology to maximize profits:* $p_t^j \frac{\partial G_t^j}{\partial k_t^{jz,d}} = r_t^z$.

Conditions (i) through (iv) equalize marginal valuations along every margin of adjustment. In (i), the representative individual equates aggregate marginal rates of substitution to goods prices. In (ii), they equate the discounted marginal revenue accumulation product of investment, $\frac{\omega_{t+1}}{\omega_t} q_{t+1}^z \frac{\partial H_{t+1}^z}{\partial \iota_t^{zj}}$, to the price of good j, p_t^j. Condition (iii) is the Euler equation governing the dynamics of capital goods prices. In (iv), firms equate the marginal revenue product of capital, $p_t^j \frac{\partial G_t^j}{\partial k_t^{jz,d}}$, to the rental rate r_t^z.

Characterizing the welfare-relevant statistics in competitive economies is critical because they determine the efficiency gains from technology change as well as the marginal welfare products behind every production efficiency component. Combining conditions (i) through (iv), it is straightforward to show that the welfare-relevant statistics in frictionless competitive interior economies are given by

$$AMRS^{j}_{t,c} = p^{j}_{t}, \quad MSV^{j}_{t,y} = MWP^{zj}_{t,\iota} = \omega_t p^{j}_{t}, \quad MSV^{z}_{t,k} = MWP^{jz}_{t,k} = \omega_t q^{z}_{t}. \tag{19}$$

That is, in frictionless competitive economies, (i) the aggregate marginal rates of substitution are given by the prices of goods, p^{j}_{t}; (ii) the marginal social value of goods and the marginal welfare product of investment are given by the discounted prices of goods, $\omega_t p^{j}_{t}$; and (iii) the marginal social value of capital goods and the marginal welfare product of capital are given by the discounted prices of capital goods, $\omega_t q^{z}_{t}$.

4.2 Production and Accumulation Hulten's Theorem

Substituting the welfare-relevant statistics (19) into the production efficiency decomposition of Theorem 1 yields the next result. Since marginal welfare products are equalized across uses — $MWP^{zj}_{t,\iota} = \omega_t p^{j}_{t}$, $\forall z$, and $MWP^{jz}_{t,k} = \omega_t q^{z}_{t}$, $\forall j$ — the cross-sectional covariance terms of Theorem 1 vanish, and since $AMWP^{j}_{t,\iota} = \omega_t AMRS^{j}_{t,c} = \omega_t p^{j}_{t}$, the aggregate investment margin vanishes as well. Production efficiency is then only due to changes in technology.

Theorem 2. *(Technology Change in Frictionless Competitive Economies) In frictionless competitive economies,*

a) (Production Efficiency) allocative efficiency does not contribute to production efficiency, which is given by

$$\Xi^{E,P} = \underbrace{\sum_t \omega_t \sum_z q^{z}_{t} \frac{\partial H^{z}_{t}}{\partial \theta}}_{\substack{\text{Accumulation} \\ \text{Technology Change}}} + \underbrace{\sum_t \omega_t \sum_j p^{j}_{t} \frac{\partial G^{j}_{t}}{\partial \theta}}_{\substack{\text{Production} \\ \text{Technology Change}}};$$

b) (Production and Accumulation Hulten's Theorem) taking date-t aggregate consumption as date-t welfare numeraire and perpetual aggregate consumption as lifetime welfare numeraire, the welfare gains from a one-percent Hicks-neutral change at date t in the production technology of good j and in the accumulation *technology of capital good z are respectively given by*

$$\omega_t \mathcal{D}^j_{t,y} \quad \textit{and} \quad \omega_t \mathcal{D}^z_{t,k},$$

where production and accumulation Domar weights are given by

$$(\textit{Production Domar Weights}) \qquad \mathcal{D}^j_{t,y} = \frac{p^j_t y^j_t}{\sum_j p^j_t c^j_t}$$

$$(\textit{Accumulation Domar Weights}) \qquad \mathcal{D}^z_{t,k} = \frac{q^z_t k^z_t}{\sum_j p^j_t c^j_t}.$$

Theorem 2a) exploits the First Welfare Theorem. Since the competitive allocation is efficient, reallocating investment or capital across uses at the margin can have no welfare effect: the covariance and aggregate-investment components of Theorem 1 are identically zero. All production efficiency gains originate in primitive changes in technologies, valued at equilibrium prices — an additional unit of capital good z is worth $\omega_t q^z_t$, and an additional unit of good j is worth $\omega_t p^j_t$, in units of the lifetime welfare numeraire. This result explains why the value of capital is the central object of this section: the entire welfare impact of a change in accumulation technologies is mediated by the prices of capital goods.

Theorem 2b) restates part a) in proportional terms and extends Hulten's theorem (Hulten, 1978) to accumulation technologies.[14] The welfare impact of a proportional improvement in a production technology equals its production Domar weight, the value of its gross output relative to nominal consumption expenditures. The new result is

[14] Formally, one can explicitly introduce production and accumulation technology shifters $\vartheta^j_{t,y}$ and $\vartheta^z_{t,k}$, so that

$$y^j_t = G^j_t\left(\left\{k^{jz,d}_t\right\}_{z\in\mathcal{Z}}; \vartheta^j_{t,y}\right) \quad \text{and} \quad k^{z,s}_{t+1} = H^z_{t+1}\left(\left\{k^{jz,d}_t\right\}_{j\in\mathcal{J}}, \left\{\iota^{zj}_t\right\}_{j\in\mathcal{J}}; \vartheta^z_{t+1,k}\right),$$

in which case production efficiency is given by

$$\Xi^{E,P} = \underbrace{\sum_t \omega_t \sum_j \mathcal{D}^j_{t,y} \frac{\partial \ln G^j_t}{\partial \ln \vartheta^j_{t,y}} \frac{d \ln \vartheta^j_{t,y}}{d\theta}}_{\substack{\text{Production} \\ \text{Technology Change}}} + \underbrace{\sum_t \omega_t \sum_z \mathcal{D}^z_{t,k} \frac{\partial \ln H^z_t}{\partial \ln \vartheta^z_{t,k}} \frac{d \ln \vartheta^z_{t,k}}{d\theta}}_{\substack{\text{Accumulation} \\ \text{Technology Change}}},$$

which covers arbitrary technology changes. A Hicks-neutral change at date t corresponds to $\frac{\partial \ln G^j_t}{\partial \ln \vartheta^j_{t,y}} = \frac{\partial \ln H^z_t}{\partial \ln \vartheta^z_{t,k}} = 1$, and a one-percent change to $\frac{d \ln \vartheta^j_{t,y}}{d\theta} = \frac{d \ln \vartheta^z_{t,k}}{d\theta} = 1$.

the accumulation counterpart: the welfare impact of a proportional improvement in an accumulation technology equals its accumulation Domar weight, the value of the corresponding capital stock relative to nominal consumption expenditures. In both cases, equilibrium prices are sufficient: no knowledge of the production network or of the accumulation technologies themselves is required to value technology changes. This formula allows us to compute the welfare impact of any change in technology, in production or in accumulation, at any date.

Remark 7. (Capital Prices as Sufficient Statistics for Changes in Accumulation Technologies). While both production and accumulation Hulten's theorems rely on prices, they do so in different ways. Production Domar weights value gross output at goods prices, p_t^j, while accumulation Domar weights value capital stocks at capital prices, q_t^z. The key insight of Theorem 2b) is that the date-t price of capital is sufficient to value the contribution of new capital, even though this capital subsequently generates both future goods and additional capital through the accumulation process. The reason is that the capital price q_t^z already capitalizes the entire sequence of future propagation. A single date-t price therefore suffices to value the technological change. To our knowledge, this result has not previously been formalized. It highlights capital markets as a central determinant of the social value of technological innovations.

4.3 Transitory vs. Permanent Technology Changes

Having valued individual technology changes, we now examine how welfare gains depend on the persistence of the change: whether it is transitory or permanent. Consider a one-percent Hicks-neutral change taking place at date τ, applied either to the production technology of good j or to the accumulation technology of capital good z, and either transitory — affecting date τ only — or permanent — affecting all dates $t \geq \tau$. Theorem 2b) characterizes the answer for all four cases.

Corollary 2. *(Hicks-Neutral Technology Changes) In frictionless competitive economies, the welfare gains from a one-percent Hicks-neutral change in the production technology of good j or in the accumulation technology of capital good z at date τ are — depending on whether the change is transitory or permanent — given by*

	Transitory	*Permanent*
Production	$\mathcal{D}^{j}_{\tau,y}$	$\sum_{t\geq\tau}\frac{\omega_t}{\omega_\tau}\mathcal{D}^{j}_{t,y}$
Accumulation	$\mathcal{D}^{z}_{\tau,k}$	$\sum_{t\geq\tau}\frac{\omega_t}{\omega_\tau}\mathcal{D}^{z}_{t,k}$

Consider first a transitory improvement in a production technology. Output rises at date τ, and no effect remains afterward: the welfare gain is the date-τ production Domar weight, and the persistence of the welfare gain mirrors the persistence of the change — a permanent improvement adds the discounted production Domar weights of all subsequent dates.

Consider next a transitory improvement in an accumulation technology. The improvement itself lasts a single period, but it delivers additional capital, and capital is durable: the additional units provide productive services at every subsequent date, decaying only at the rate of depreciation. The welfare gain is nonetheless captured by a single date-τ statistic, the accumulation Domar weight, because the price of capital capitalizes the entire stream of rentals the additional capital will earn. A permanent improvement in an accumulation technology can in turn be interpreted as a sequence of transitory improvements, one at each subsequent date, each with a persistent welfare effect of its own. Permanent changes in accumulation technologies can consequently have large welfare effects. The discounted sum of accumulation Domar weights, $\sum_{t\geq\tau}\frac{\omega_t}{\omega_\tau}\mathcal{D}^{z}_{t,k}$, prices exactly this process: each term already capitalizes an entire stream of future rentals, and the sum adds these capitalized streams across all dates $t \geq \tau$.

4.4 Domar Sums

Summing the transitory entries of Corollary 2 across goods and across capital goods yields the *production Domar sum*, $\sum_j \mathcal{D}^{j}_{t,y}$, and the *accumulation Domar sum*, $\sum_z \mathcal{D}^{z}_{t,k}$: the welfare gains from a one-percent Hicks-neutral change applied to all production or to all accumulation technologies at date t. There exists a clear relation between the two sums, and comparing them reveals whether across-the-board production or accumulation technology shocks carry higher value.[15]

[15] To simplify the exposition, we assume constant-returns-to-scale in production for this section, which ensures that profits are zero.

Corollary 3. *(Domar Weight Ordering) In frictionless competitive economies, at every date $t < T$ at which investment is positive and capital does not fully depreciate*

$$\underbrace{\sum_z \mathcal{D}^z_{t,k}}_{\substack{\text{Accumulation}\\ \text{Domar Sum}}} \geq \underbrace{\sum_j \mathcal{D}^j_{t,y}}_{\substack{\text{Production}\\ \text{Domar Sum}}} \geq 1.$$

The proof of Corollary 3 is a chain of valuations:

$$\underbrace{\sum_j p^j_t c^j_t}_{\text{Consumption}} < \underbrace{\sum_j p^j_t y^j_t}_{\text{Gross Output}} = \underbrace{\sum_z r^z_t k^z_t}_{\text{Capital Rentals}} < \underbrace{\sum_z q^z_t k^z_t}_{\text{Value of Capital}} .$$

The first gap is the value of investment: consumption is the only final good, and investment goods are intertemporal intermediate inputs, so gross output exceeds consumption expenditures — just as intermediate inputs push the standard Domar sum above one. The second gap is the stock-flow distinction: rental payments already account for the value of gross output, and the price of installed capital adds the continuation value on top, since capital is valued for the output it yields today and for the undepreciated stock it carries into the future. At the terminal date, both gaps close: investment is zero in any interior equilibrium — it reduces consumption with no offsetting benefit — and the price of capital equals its rental rate, so both sums equal one.

Formally, multiplying the capital Euler equations — condition (iii) — by $k^{jz,s}_t$ and summing across pairs yields

$$\sum_z q^z_t k^z_t = \sum_z r^z_t k^z_t + \mathcal{C}_t, \quad \text{where} \quad \mathcal{C}_t = \frac{\Lambda_{t+1}}{\Lambda_t} \sum_z q^z_{t+1} \sum_j \frac{\partial H^z_{t+1}}{\partial k^{jz,s}_t} k^{jz,s}_t \geq 0.$$

The continuation value $\mathcal{C}_t$ is the discounted value of the capital that survives into date $t+1$, and Λ_t denotes the Lagrange multiplier associated with the date-t budget constraint. When accumulation technologies also exhibit constant returns to scale, iterating this identity forward yields $\Lambda_t \sum_z q^z_t k^z_t = \sum_{s\geq t} \Lambda_s \sum_j p^j_s c^j_s$: the value of capital equals the present value of consumption, so the accumulation Domar sum satisfies $\sum_z \mathcal{D}^z_{t,k} = \sum_{s\geq t} \frac{\omega_s}{\omega_t}$.

Remark 8. *(Beyond the Baseline Environment)* The unambiguous ordering of the Domar sums in Corollary 3 can in principle be reversed when production technologies also use intermediate inputs and additional factors. As shown in the Appendix, in this case, the gross output identity becomes $\sum_j p_t^j y_t^j = \sum_z r_t^z k_t^z + \sum_f w_t^f n_t^{f,d} + \sum_j p_t^j x_t^j$, and the difference between the Domar sums becomes

$$\sum_z \mathcal{D}_{t,k}^z - \sum_j \mathcal{D}_{t,y}^j = \frac{\mathcal{C}_t - \left(\sum_f w_t^f n_t^{f,d} + \sum_j p_t^j x_t^j\right)}{\sum_j p_t^j c_t^j},$$

so the accumulation Domar sum exceeds the production Domar sum if and only if the continuation value of capital exceeds date-t payments to factors and intermediate inputs. Equivalently, in ratio form,

$$\frac{\sum_z \mathcal{D}_{t,k}^z}{\sum_j \mathcal{D}_{t,y}^j} = \underbrace{\frac{\sum_z r_t^z k_t^z}{\sum_j p_t^j y_t^j}}_{\substack{\text{Capital Share of} \\ \text{Gross Output}} \leq 1} \times \underbrace{\left(1 + \frac{\mathcal{C}_t}{\sum_z r_t^z k_t^z}\right)}_{\substack{\text{Capitalization} \\ \text{Factor}} \geq 1}.$$

The first term falls below one when payments to other factors or intermediate inputs are positive; the second always exceeds one when capital survives. Since the two terms push in opposite directions, whether the accumulation Domar sum exceeds the production Domar sum is an empirical question, which we study next.

5 Application 1: Measuring Accumulation and Production Domar Weights

Motivated by Theorem 2 and Corollaries 2 and 3, we now proceed to measure and analyze accumulation and production Domar weights in the U.S. economy and their evolution. An interpretation of the finding that accumulation weights are larger than production weights is that technological innovations to accumulation can be more valuable than those to production.

Data Description. We focus on the postwar U.S. economy between 1947 and 2024. For each of the 20 2-digit North American Industry Classification System (NAICS) industries, we observe nominal gross output and aggregate nominal final consumption from the BEA input-output (I-O) Use tables. This allows us to compute production Domar weights at each date. For each of those 20 industries, we also observe nominal capital from the BEA's Fixed Asset tables, split into three categories: (i) Equipment, (ii) Structures, and (iii) Intellectual Property Products (IPP). This allows us to compute accumulation Domar weights aggregated and disaggregated per category per industry.[16] When needed to compute ω_t, we assume an isoelastic flow utility such that $V = \sum_t \beta^t \frac{C_t^{1-\gamma}}{1-\gamma}$ where $C_t = \left[\sum_{j=1}^{J} \left(c_t^j\right)^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}}$ is a CES consumption aggregator. We use standard values: $\beta = 0.98$, $\gamma = 2$, and $\sigma = 0.349$ as in Schaab and Tan (2025). Section D.1 of the Online Appendix includes a more detailed description of the data.

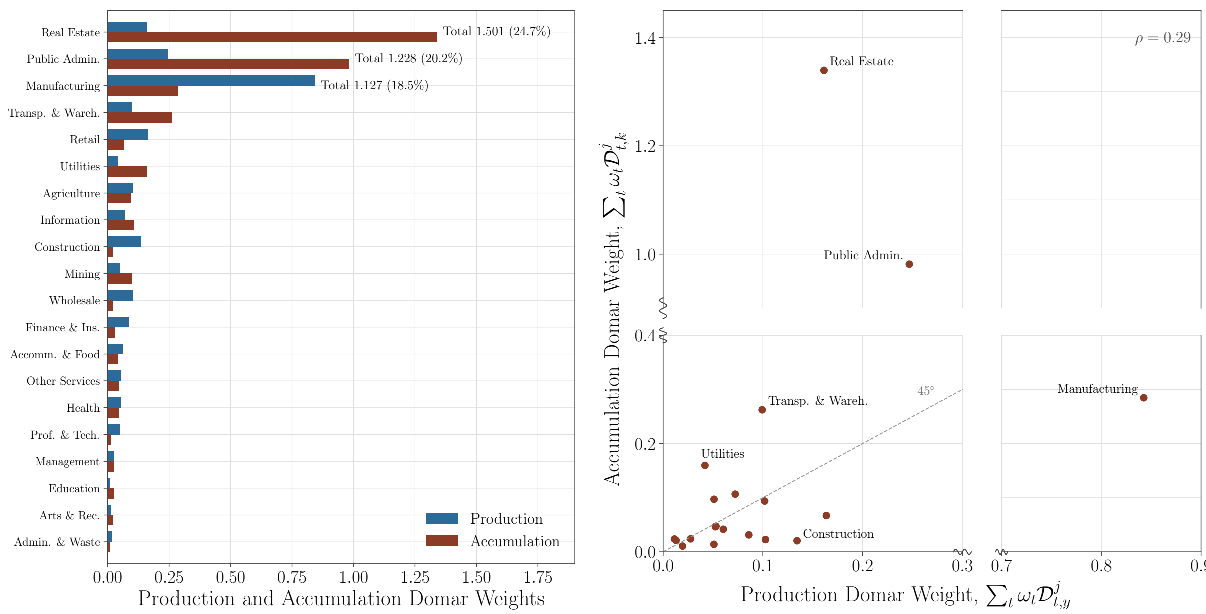


Figure 2: Application 1: Measuring Domar Weights

Note: Discounted sums of production and accumulation Domar weights, $\sum_t \omega_t \mathcal{D}_{t,y}^j$ and $\sum_t \omega_t \mathcal{D}_{t,k}^j$, by industry — the welfare gains from permanent one-percent Hicks-neutral technology changes — under the lifetime welfare numeraire $\lambda = \sum_t \lambda_t$ (postwar U.S., 1947–2024; $\beta = 0.98$, $\gamma = 2$, $\sigma = 0.349$). Data on nominal gross output and final consumption are taken from the BEA I-O Use tables and data on nominal capital are taken from the BEA's Fixed Asset tables. Industries are sorted by total contribution in the left panel; the three largest are annotated with their totals and shares of the aggregate 6.087. The right panel plots each industry's discounted sum of accumulation Domar weights against its production counterpart; the dashed line is the 45-degree line and ρ denotes the correlation.

Findings: Industry Weights. The left panel of Figure 2 reports the discounted production and accumulation Domar weights by industry, $\sum_t \omega_t \mathcal{D}_{t,y}^j$ and $\sum_t \omega_t \mathcal{D}_{t,k}^j$. These can be interpreted as time-averages of Domar weights and, by Corollary 2, as the welfare gains from permanent one-percent Hicks-neutral improvements in each industry's

[16]Hence, we implicitly map the 20 2-digit industries to J, and then assume that $Z = 3J$ when reporting disaggregated numbers, or $Z = J$ when aggregated.

production and accumulation technologies. Summing across industries, a permanent one-percent improvement in all technologies yields welfare gains of 6.087% of lifetime consumption, with 2.391 percentage points originating in production Domar weights and 3.695 in accumulation Domar weights (roughly 61%).

Weights are significantly concentrated, as shown in the left panel of Figure 2. Real Estate and Rental and Leasing alone accounts for 1.501 (24.7% of the total), Public Administration accounts for 1.228 (20.2%), and Manufacturing accounts for 1.127 (18.5%); together these three industries contribute 63.4% of the total. Industries also differ in the composition of their contribution: accumulation Domar weights dominate for Real Estate and Public Administration, whereas production Domar weights dominate for Manufacturing. Transportation and Warehousing (0.362), Retail Trade (0.231), Utilities (0.202), Agriculture (0.196), Information (0.179), and Construction (0.155) follow. Table OA2 in the Appendix reports the exact values. The right panel of Figure 2 plots each industry's discounted sum of accumulation Domar weights against its production counterpart: Real Estate and Public Administration lie far above the 45-degree line, whereas Manufacturing lies far below it.

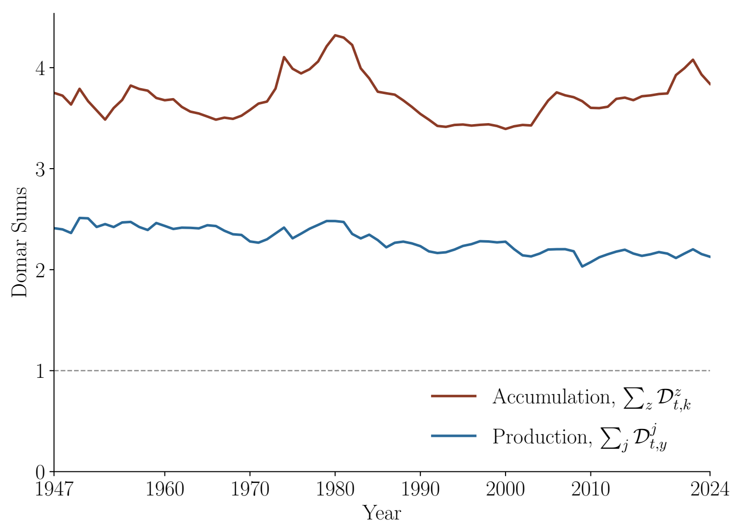


Figure 3: Application 1: Evolution of Domar Sums

Note: Accumulation and production Domar sums, $\sum_z \mathcal{D}^z_{t,k}$ and $\sum_j \mathcal{D}^j_{t,y}$, computed year by year (postwar U.S., 1947–2024; $\beta = 0.98$, $\gamma = 2$, $\sigma = 0.349$). The dashed line marks 1.

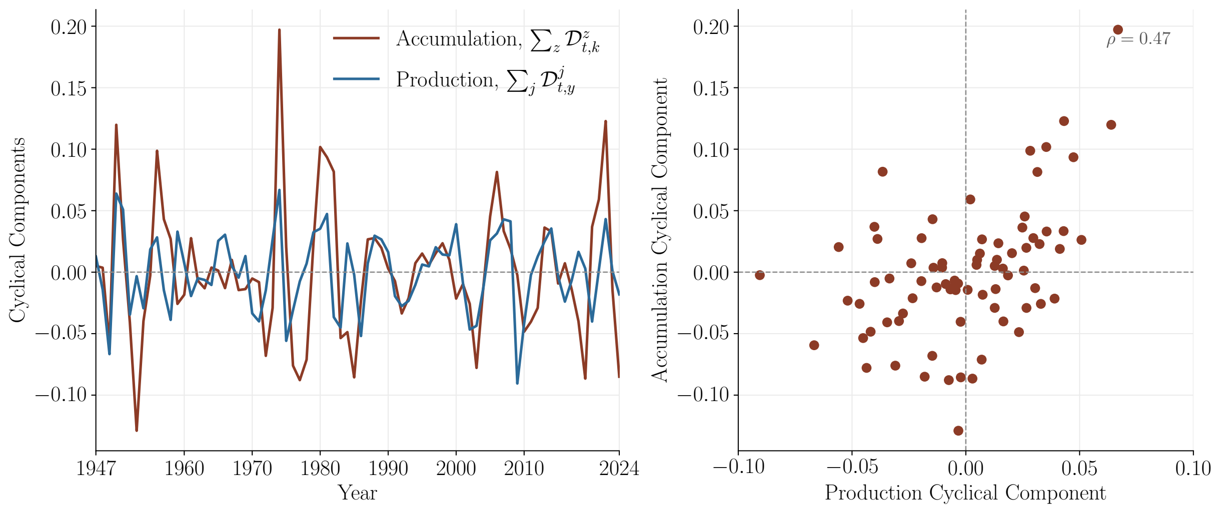


Figure 4: Application 1: Domar Sums Cyclicality

Note: Accumulation and production Domar sums, $\sum_z \mathcal{D}^z_{t,k}$ and $\sum_j \mathcal{D}^j_{t,y}$ (postwar U.S., 1947–2024; $\beta = 0.98$, $\gamma = 2$, $\sigma = 0.349$). The left panel reports the Hodrick–Prescott cyclical components; the right panel illustrates their comovement, with ρ denoting the correlation between the two cyclical components.

Findings: Domar Sums. Figure 3 plots the accumulation and production Domar sums, $\sum_z \mathcal{D}^z_{t,k}$ and $\sum_j \mathcal{D}^j_{t,y}$, year by year. The Domar weight ordering of Corollary 3, $\sum_z \mathcal{D}^z_{t,k} > \sum_j \mathcal{D}^j_{t,y} > 1$, thus holds in every year of the sample: the accumulation sum fluctuates between 3.4 and 4.4 — peaking around 1980 — while the production sum remains between 2.0 and 2.5, so their ratio ranges from 1.4 to 1.9. This finding indicates that technological innovations to accumulation can be more valuable than those to production.

Figure 4 separates each sum into a Hodrick–Prescott trend and a cyclical component: the accumulation sum is substantially more cyclical than the production sum, while the two cyclical components strongly comove positively.

Additional Findings. We report several additional findings in Appendix D.1. Three are worth highlighting. First, time variation in the industry-level weights is minimal for the discounted sums: replacing each industry's Domar weight path by its unweighted time mean leaves the discounted sum of accumulation Domar weights essentially unchanged and lowers its production counterpart by 4.1%, so Figure 2 should be read as a description of the ω_t-weighted postwar path, dominated by the early decades. Second, the industry-level series reallocate substantially over the postwar period: Real Estate and Public

Administration carry the two largest accumulation Domar weights throughout, while production Domar weights decline in Manufacturing and rise in service industries (Figure OA4). Third, Structures absorb 79% of the discounted sum of accumulation Domar weights (Table OA1). Accordingly, the accumulation Domar weights of the U.S. economy are dominated by Structures.

6 Application 2: Origins of Welfare Losses in a Credit Crunch

This application identifies the origins of welfare losses associated with a credit crunch. A large literature studies the output losses associated with financial crises (Buera, Kaboski, and Shin, 2011; Moll, 2014; Buera and Moll, 2015). Output-based measures, however, cannot determine whether the associated welfare losses stem from misallocating capital across sectors, from accumulating too little capital in the aggregate, or from reallocating resources between constrained and unconstrained individuals. Theorem 1 separates these margins. A credit crunch is therefore an ideal setting to put the results to use, as a change in the tightness of a borrowing/collateral constraint operates through multiple margins at once, and we are able to quantify the contribution of each.

Environment. We consider a deterministic economy with $I = 2$ individuals, two intermediate goods, one final good, and a single capital good. Time is discrete, with dates indexed by $t \in \{0, \dots, T\}$, where $T \leq \infty$. The environment is a two-individual variant of Banerjee and Moll (2010); the credit-crunch experiment follows Buera and Moll (2015). Intermediate good $i = 1$ is produced by services, a small-scale industry, and intermediate good $i = 2$ by manufacturing, a large-scale industry.[17] Guided by the evidence on establishment scale, we model services as the unconstrained sector and manufacturing as the constrained sector.[18]

[17] Note that to simplify the exposition, we are implicitly mapping $j = \{1, 2\}$ to $i = \{1, 2\}$.

[18] Scale is the defining sectoral characteristic in Buera, Kaboski, and Shin (2011): manufacturing operates at roughly three times the establishment scale of services, and larger scale means larger per-establishment financing needs, so financial frictions fall hardest on manufacturing. Mirroring their mechanism, we assign the collateral constraint to the manufacturing industry.

Individual i derives utility from consuming the final good,

$$V^i = \sum_{t=0}^{T} \left(\beta^i\right)^t u\left(c_t^i\right) \quad \text{where} \quad u(c) = \frac{c^{1-\gamma}}{1-\gamma}.$$

Using the final good as numeraire, individual i's budget constraint is

$$c_t^i + a_{t+1}^i = (r_t + 1 - \delta)\, a_t^i + \pi_t^i,$$

where a_t^i denotes individual i's wealth, r_t the rental rate of the capital good, and π_t^i the profits of production technology i.

Intermediate good i is produced by a firm owned by individual i that rents capital and maximizes date-t profits subject to a collateral constraint tied to its owner's wealth,

$$\pi_t^i = \max_{k_t^{i,d}} p_t^i y_t^i - r_t k_t^{i,d} \quad \text{s.t.} \quad y_t^i = \left(k_t^{i,d}\right)^\alpha \quad \text{and} \quad k_t^{i,d} \leq \frac{a_t^i}{\theta},$$

where $\theta \in [0, 1]$ indexes the tightness of the credit market: a higher θ raises the wealth required per unit of capital rented. The final good is produced with the constant-returns Cobb–Douglas technology

$$y_t = \prod_{i=1}^{2} \left(x_t^i\right)^{\zeta_y^i},$$

where $\zeta_y^i \geq 0$ and $\sum_{i=1}^{2} \zeta_y^i = 1$.[19] The single accumulation technology is given by

$$k_{t+1} = (1-\delta) \sum_i k_t^{i,d} + \iota_t.$$

Finally, the resource constraints are

$$y_t = c_t + \iota_t, \quad y_t^i = x_t^i, \quad \text{and} \quad k_t = \sum_i k_t^{i,d} = k_t^d,$$

where $c_t = \sum_i c_t^i$.

Origins of Efficiency Losses. We consider the welfare effects of the credit-market perturbation for a utilitarian planner, $W = \sum_i V^i$. We use date-t aggregate consumption

[19] The unit-elastic aggregator follows Buera, Kaboski, and Shin (2011), who set the elasticity of substitution between services and manufacturing to one and argue that the choice is conservative.

as date-t welfare numeraire and perpetual aggregate consumption as lifetime welfare numeraire,

$$\lambda_t^i = \left(\beta^i\right)^t u'\left(c_t^i\right) c_t \quad \text{and} \quad \lambda^i = \sum_t \lambda_t^i,$$

so that $MRS_{t,c}^i = \frac{\left(\beta^i\right)^t u'\left(c_t^i\right)}{\lambda_t^i} = \frac{1}{c_t}$, which in turn implies that $MRS_{t,c}^i$ is equalized across individuals and exchange efficiency vanishes. The welfare decomposition is given by

$$\frac{dW^\lambda}{d\theta} = \underbrace{\sum_t AMRS_{t,c} \frac{dc_t}{d\theta}}_{\Xi^{AE} \text{ (Aggregate Efficiency)}} + \underbrace{\sum_t \omega_t \mathbb{C}ov_i^\Sigma \left[\frac{\omega_t^i}{\omega_t}, \frac{MRS_{t,c}^i}{\omega_t^i} \frac{dc_t^i}{d\theta}\right]}_{\Xi^{IS} \text{ (Intertemporal-Sharing)}} + \underbrace{\mathbb{C}ov_i^\Sigma \left[\omega^i, \frac{dV^{i|\lambda}}{d\theta}\right]}_{\Xi^{RD} \text{ (Redistribution)}},$$

where the normalized individual weights, dynamic weights, and average dynamic weights are

$$\omega^i = \frac{\lambda^i}{\frac{1}{I}\sum_i \lambda^i}, \quad \omega_t^i = \frac{\lambda_t^i}{\lambda^i}, \quad \text{and} \quad \omega_t = \frac{1}{I}\sum_i \omega_t^i,$$

and $\mathbb{C}ov_i^\Sigma [\cdot,\cdot] = I \cdot \mathbb{C}ov_i [\cdot,\cdot]$ denotes a covariance-sum across individuals.

Transfers between constrained and unconstrained individuals affect welfare through two distinct components. Their efficiency content appears as intertemporal-sharing: constrained and unconstrained individuals value date-t resources differently relative to their lifetime valuations — their normalized dynamic weights ω_t^i differ — so transferring resources toward the individual who values them relatively more at a given date increases efficiency. Their equity content appears as redistribution, which captures the equity concerns embedded in the social welfare function through the covariance between the normalized individual weights ω^i and individual lifetime welfare changes.

Because $MRS_{t,c}^i$ are equalized, aggregate efficiency involves only production efficiency. With a single capital good, cross-sectional investment efficiency vanishes, and absent technology changes the decomposition reduces to two possibly nonzero components,

$$\Xi^{AE} = \Xi^{AE,P} = \underbrace{\sum_t \mathbb{C}ov_j^\Sigma \left[MWP_{t,k}^j, \frac{d\chi_{t,k}^{j,d}}{d\theta}\right] k_t^d}_{\substack{\text{Cross-Sectional} \\ \text{Capital Use Efficiency}}} + \underbrace{\sum_t \left(AMWP_{t,\iota} - \omega_t AMRS_{t,c}\right) \frac{d\phi_{t,\iota}}{d\theta} y_t}_{\substack{\text{Aggregate} \\ \text{Investment Efficiency}}}.$$

Calibration. Table 2 summarizes our parametrization. We set $\gamma = 2$ and take the remaining parameters from postwar U.S. data (1947–2024): $\alpha = 0.4505$ is one minus

Table 2: Parameter Values (Application 2)

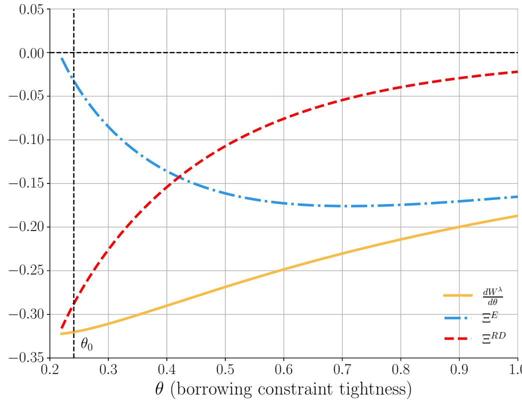
	Description	Parameter(s)	Value(s)
Preferences	Utility Curvature	γ	2
	Discount Factors	β^1, β^2	$0.9937, 0.9390$
Technology	Capital Share	α	0.4505
	Depreciation Rate	δ	0.0483
	Final-Good Production Shares	ζ_y^1, ζ_y^2	$0.4915, 0.5085$
Credit Market	Collateral Parameter	θ_0	0.2410
Initial Conditions	Initial Wealth	a_0^1, a_0^2	$19.124, 2.332$

Note: This table summarizes the choice of parameter values used in Section 6.

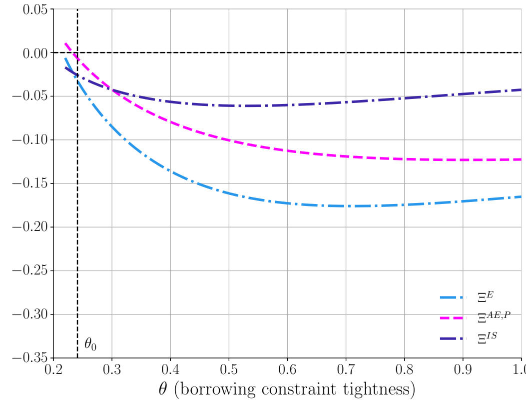
the average aggregate labor share, $\delta = 0.0483$ the average depreciation rate, and $r^\star = 0.0546$ the average real interest rate, so that $\beta^1 = \frac{1}{1+r^\star-\delta} = 0.9937$; $\zeta_y^1 = 0.4915$ and $\zeta_y^2 = 0.5085$ are services' share of total nominal gross output and its complement. The credit-market parameters are disciplined by external finance: $\theta_0 = \frac{1}{4.15} = 0.241$ adopts the U.S. collateral parameter of Moll (2014), and $\beta^2 = 0.9390$ is calibrated so that steady-state external finance matches the U.S. external-finance-to-GDP ratio of Beck, Demirgüç-Kunt, and Levine (2000). We start every transition from the steady state, $a_0 = a^\star(\theta_0) = (19.124, 2.332)$.

Results. We model a credit crunch as an increase in θ, that is, a tightening of the collateral constraint $k_t^{i,d} \leq \frac{a_t^i}{\theta}$, so the experiment is $d\theta > 0$. A marginal credit crunch reduces utilitarian welfare at every level of tightness. Near the status quo $\theta_0 = 0.241$, the efficiency loss is small, and driven by both intertemporal-sharing considerations (as the initially constrained individuals can borrow even less) and production efficiency. The same forces increase the marginal efficiency loss as the constraint keeps tightening.

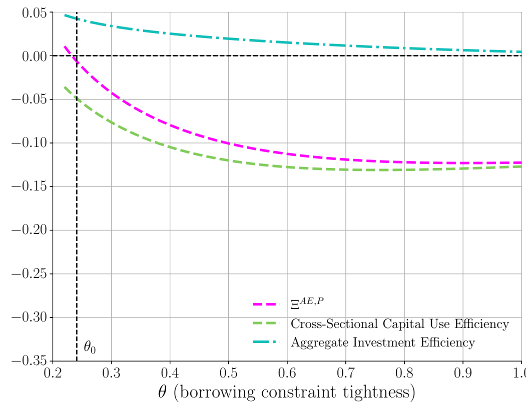
The split of production efficiency losses is somewhat surprising. First, at all times, tightening the borrowing constraint generates losses due to capital misallocation, as captured by the cross-sectional capital use efficiency term. However, this tightening is associated with a positive aggregate investment effect. At each date, a marginal unit of the final good is worth $\omega_t AMRS_{t,c}$ if consumed and $AMWP_{t,\iota}$ if invested. In a frictionless equilibrium, individuals equalize both margins, but with collateral constraints, the constrained individual's private return to saving includes the collateral shadow value



(a) Welfare Decomposition



(b) Efficiency Decomposition



(c) Production Efficiency Decomposition

Figure 5: Application 2: Origins of Welfare Losses in a Credit Crunch

Note: Panel (a): welfare accounting decomposition for a utilitarian planner. Panel (b): efficiency and its two components, aggregate efficiency and intertemporal-sharing. Panel (c): production efficiency and its two components, cross-sectional capital use efficiency and aggregate investment efficiency. All objects are in units of perpetual aggregate consumption.

— every unit of wealth relaxes tomorrow's constraint — so there is too much investment in equilibrium, leaving the marginal invested unit socially worth less than the marginal consumed unit ($AMWP_{t,\iota} < \omega_t AMRS_{t,c}$). Because a credit crunch lowers the investment share at essentially every date ($d\phi_{t,\iota}/d\theta < 0$), it moves final goods from investment to consumption and each shifted unit yields a positive welfare gain on this margin.

Since the tightening of the borrowing constraints already hurts the borrowing-constrained individuals, an equal-weighted utilitarian planner perceives an additional redistribution loss, as shown in panel (a) of Figure 5.

7 Conclusion

This paper systematically identifies the origins of welfare gains and losses in economies with accumulation technologies, in which capital links resource constraints across dates. It draws three main conclusions.

First, welfare gains coincide with production efficiency gains and originate through five channels: cross-sectional and aggregate investment efficiency, cross-sectional capital use efficiency, and accumulation and production technology change. A defining feature of our approach is that it is exclusively based on preferences, technologies, and resource constraints, without invoking prices, budget constraints, or equilibrium notions. It therefore identifies the origins of the welfare gains from any perturbation of any feasible allocation — efficient or not — under any institutional or market structure.

Second, in frictionless competitive economies all allocative efficiency components vanish, and the value of assets becomes the key object for the welfare assessment of technological change. We show that the welfare gain from a one-percent improvement in a production technology equals its production Domar weight — the well-known production Hulten's theorem — while the welfare gain from a one-percent improvement in an accumulation technology equals its accumulation Domar weight: the value of the corresponding capital stock relative to nominal consumption expenditures — a new accumulation Hulten's theorem. A single date-t capital price is sufficient to value a technology change whose effects extend over every subsequent date, since the price of installed capital already capitalizes the entire sequence of future propagation through production and accumulation. At every date, the accumulation Domar sum (the sum of all accumulation Domar weights) weakly exceeds the production Domar sum, which weakly exceeds one.

Third, the decomposition can be taken directly to the data and to quantitative models. Between 1947 and 2024, the accumulation Domar sum exceeds the production Domar sum every year, with a ratio between 1.4 and 1.9. Our measures imply that a permanent one-percent Hicks-neutral improvement in all technologies would yield welfare gains of 6%, with roughly 61% originating in accumulation technologies, suggesting that technological innovations to accumulation can be more valuable than technological innovations to production. In a credit crunch, our approach allows us to separate the welfare losses due to misallocating capital across producers from the effects of changing aggregate accumulation and from transfers between constrained and unconstrained individuals, who value consumption differently at different dates. Capital misallocation and reallocating resources away from already constrained individuals generate losses at every level of the collateral constraints, while the aggregate investment margin generates a gain: collateral constraints induce overinvestment, so a credit crunch shifts final goods from investment toward consumption, where their social value is higher.

Our results imply that assessing the welfare impact of technological change in economies with capital critically hinges on valuing stocks of assets in addition to flows of output. As long as national accounts measure capital stocks at the industry and asset level, accumulation Domar weights can be computed wherever production Domar weights can — across countries, time periods, and finer industry and asset categories. We hope the approach developed in this paper, which the Online Appendix extends to heterogeneous individuals, elastically supplied factors, intermediate inputs, and uncertainty, proves useful to identify the origins of welfare gains in the rich class of economies with accumulation.

References

Abel, A. B. (1981): "A Dynamic Model of Investment and Capacity Utilization," *Quarterly Journal of Economics*, 96(3), 379–403.

Abel, A. B., and J. C. Eberly (1994): "A Unified Model of Investment Under Uncertainty," *American Economic Review*, 84(5), 1369–1384.

Abel, A. B., N. G. Mankiw, L. H. Summers, and R. J. Zeckhauser (1989): "Assessing Dynamic Efficiency: Theory and Evidence," *Review of Economic Studies*, 56(1), 1–19.

Alvarez, F., and U. J. Jermann (2004): "Using Asset Prices to Measure the Cost of Business Cycles," *Journal of Political Economy*, 112(6), 1223–1256.

Atalay, E. (2017): "How Important Are Sectoral Shocks?," *American Economic Journal: Macroeconomics*, 9(4), 254–280.

Banerjee, A. V., and B. Moll (2010): "Why Does Misallocation Persist?," *American Economic Journal: Macroeconomics*, 2(1), 189–206.

Baqaee, D., and E. Farhi (2020): "Productivity and Misallocation in General Equilibrium," *Quarterly Journal of Economics*, 135(1), 105–163.

Baqaee, D., and H. Malmberg (2025): "Long-Run Comparative Statics," *NBER Working Paper*.

Barro, R. J. (2021): "Double Counting of Investment," *Economic Journal*, 131(638), 2333–2356.

Basu, S., and J. G. Fernald (2002): "Aggregate Productivity and Aggregate Technology," *European Economic Review*, 46(6), 963–991.

Basu, S., L. Pascali, F. Schiantarelli, and L. Servén (2022): "Productivity and the Welfare of Nations," *Journal of the European Economic Association*, 20(4), 1647–1682.

Beck, T., A. Demirgüç-Kunt, and R. Levine (2000): "A New Database on the Structure and Development of the Financial Sector," *World Bank Economic Review*, 14(3), 597–605.

Berger, D., L. Bocola, and A. Dovis (2023): "Imperfect Risk Sharing and the Business Cycle," *Quarterly Journal of Economics*, 138(3), 1765–1815.

Brunnermeier, M. K., and Y. Sannikov (2014): "A Macroeconomic Model with a Financial Sector," *American Economic Review*, 104(2), 379–421.

Buera, F. J., J. P. Kaboski, and Y. Shin (2011): "Finance and Development: A Tale of Two Sectors," *American Economic Review*, 101(5), 1964–2002.

Buera, F. J., and B. Moll (2015): "Aggregate Implications of a Credit Crunch: The Importance of Heterogeneity," *American Economic Journal: Macroeconomics*, 7(3), 1–42.

Carvalho, V. M., M. Covarrubias, and G. Nuño (2026): "Planning against Disasters in

Dynamic Production Networks," *Working Paper.*

Cass, D. (1965): "Optimum Growth in an Aggregative Model of Capital Accumulation," *Review of Economic Studies*, 32(3), 233–240.

——— (1972): "On Capital Overaccumulation in the Aggregative, Neoclassical Model of Economic Growth: A Complete Characterization," *Journal of Economic Theory*, 4(2), 200–223.

Chari, V. V., P. J. Kehoe, and E. R. McGrattan (2007): "Business Cycle Accounting," *Econometrica*, 75(3), 781–836.

David, J. M., and V. Venkateswaran (2019): "The Sources of Capital Misallocation," *American Economic Review*, 109(7), 2531–2567.

Dávila, E., P. Jeenas, and T. Philippon (2026): "Incompleteness Shocks," *Working Paper.*

Dávila, E., and A. Schaab (2023): "Welfare Accounting," *NBER Working Paper.*

——— (2025): "Welfare Assessments with Heterogeneous Individuals," *Journal of Political Economy*, 133(9), 2918–2961.

Eisfeldt, A. L., and A. A. Rampini (2008): "Managerial Incentives, Capital Reallocation, and the Business Cycle," *Journal of Financial Economics*, 87(1), 177–199.

Fisher, J. D. M. (2006): "The Dynamic Effects of Neutral and Investment-Specific Technology Shocks," *Journal of Political Economy*, 114(3), 413–451.

Foerster, A. T., A. Hornstein, P.-D. G. Sarte, and M. W. Watson (2022): "Aggregate Implications of Changing Sectoral Trends," *Journal of Political Economy*, 130(12), 3286–3333.

Foerster, A. T., P.-D. G. Sarte, and M. W. Watson (2011): "Sectoral versus Aggregate Shocks: A Structural Factor Analysis of Industrial Production," *Journal of Political Economy*, 119(1), 1–38.

Fuchs, W., B. Green, and D. Papanikolaou (2016): "Adverse Selection, Slow-Moving Capital, and Misallocation," *Journal of Financial Economics*, 120(2), 286–308.

Gaillard, A., W. Morrison, and P. Wangner (2026): "The Savings Wedge," *Working Paper.*

Greenwood, J., Z. Hercowitz, and P. Krusell (1997): "Long-Run Implications of Investment-Specific Technological Change," *American Economic Review*, 87(3), 342–362.

Hall, R. E. (1990): "Invariance Properties of Solow's Productivity Residual," in *Growth/Productivity/Unemployment: Essays to Celebrate Bob Solow's Birthday*, ed. by P. Diamond, pp. 71–112. MIT Press.

Hayashi, F. (1982): "Tobin's Marginal q and Average q: A Neoclassical Interpretation," *Econometrica*, 50(1), 213–224.

Horvath, M. (2000): "Sectoral Shocks and Aggregate Fluctuations," *Journal of Monetary*

Economics, 45(1), 69–106.

Hsieh, C.-T., and P. J. Klenow (2009): "Misallocation and Manufacturing TFP in China and India," *Quarterly Journal of Economics*, 124(4), 1403–1448.

Hulten, C. R. (1978): "Growth Accounting with Intermediate Inputs," *Review of Economic Studies*, 45(3), 511–518.

——— (1979): "On the "Importance" of Productivity Change," *American Economic Review*, 69(1), 126–136.

Jorgenson, D. W. (1963): "Capital Theory and Investment Behavior," *American Economic Review*, 53(2), 247–259.

Kiyotaki, N., and J. Moore (1997): "Credit Cycles," *Journal of Political Economy*, 105(2), 211–248.

Koopmans, T. C. (1965): "On the Concept of Optimal Economic Growth," in *The Econometric Approach to Development Planning*, pp. 225–287. North-Holland.

Kuznets, S. (1941): *National Income, 1919–1938.* National Bureau of Economic Research.

Kydland, F. E., and E. C. Prescott (1982): "Time to Build and Aggregate Fluctuations," *Econometrica*, 50(6), 1345–1370.

Lucas, R. E. (1987): *Models of Business Cycles.* Basil Blackwell.

——— (1988): "On the Mechanics of Economic Development," *Journal of Monetary Economics*, 22(1), 3–42.

Moll, B. (2014): "Productivity Losses from Financial Frictions: Can Self-Financing Undo Capital Misallocation?," *American Economic Review*, 104(10), 3186–3221.

Nordhaus, W. D., and J. Tobin (1973): "Is Growth Obsolete?," in *The Measurement of Economic and Social Performance*, ed. by M. Moss, pp. 509–564. NBER.

Ramsey, F. P. (1928): "A Mathematical Theory of Saving," *Economic Journal*, 38(152), 543–559.

Rebelo, S. (1991): "Long-Run Policy Analysis and Long-Run Growth," *Journal of Political Economy*, 99(3), 500–521.

Restuccia, D., and R. Rogerson (2008): "Policy Distortions and Aggregate Productivity with Heterogeneous Establishments," *Review of Economic Dynamics*, 11(4), 707–720.

Schaab, A., and S. Y. Tan (2025): "Monetary and Fiscal Policy According to HANK-IO," *Working Paper*.

Solow, R. M. (1957): "Technical Change and the Aggregate Production Function," *Review of Economics and Statistics*, 39(3), 312–320.

Tobin, J. (1969): "A General Equilibrium Approach to Monetary Theory," *Journal of Money, Credit and Banking*, 1(1), 15–29.

Uzawa, H. (1965): "Optimum Technical Change in an Aggregative Model of Economic Growth," *International Economic Review*, 6(1), 18–31.

——— (1969): "Time Preference and the Penrose Effect in a Two-Class Model of Economic Growth," *Journal of Political Economy*, 77(4, Part 2), 628–652.

vom Lehn, C., and T. Winberry (2022): "The Investment Network, Sectoral Comovement, and the Changing US Business Cycle," *Quarterly Journal of Economics*, 137(1), 387–433.

Weitzman, M. L. (1976): "On the Welfare Significance of National Product in a Dynamic Economy," *Quarterly Journal of Economics*, 90(1), 156–162.

Online Appendix

Section A of this Online Appendix provides explicit definitions of the vectors and matrices used in the paper. Section B generalizes our results in the body of the paper to account for intermediate inputs, multiple factors, and endowments, and to allow for heterogeneous individuals. Section C presents proofs and derivations. Section E includes additional results.

To simplify the exposition, we assume throughout that (i) consumption is (weakly) desirable, i.e., $\partial u_t/\partial c_t^j \geq 0$; (ii) the marginal products of capital goods are weakly positive, i.e., $\partial G_t^j/\partial k_t^{jz,d} \geq 0$; (iii) the marginal accumulation products of capital goods and investments are weakly positive, i.e., $\partial H_{t+1}^z/\partial k_t^{jz,d} \geq 0$ and $\partial H_{t+1}^z/\partial \iota_t^{zj} \geq 0$; and (iv) the no-free-lunch property holds, i.e., $G_t^j(\cdot) = 0$ if $k_t^{jz,d} = 0$, $\forall z$, and $H_{t+1}^z(\cdot) = 0$ if $k_t^{jz,d} = 0$ and $\iota_t^{zj} = 0$, $\forall j$. Many of our results, including the welfare accounting decomposition, do not require such restrictions.

A Matrix Definitions

This section defines all matrices used in the body of the paper and in this Appendix. To simplify the exposition, we represent all matrices for the $I = 2$, $J = 2$, $Z = 2$, $F = 2$ case, although we define matrix dimensions for the general case. For clarity, we typically use L to denote the number of intermediate inputs, although $L = J$.

Allocations. We collect capital good uses at date t, $k_t^{jz,d}$, in the $JZ \times 1$ vector $\mathring{\boldsymbol{k}}_t^d$, factor uses at date t, $n_t^{jf,d}$, in the $JF \times 1$ vector $\mathring{\boldsymbol{n}}_t^d$, and investment allocations at date t, ι_t^{zj}, in a $ZJ \times 1$ vector $\mathring{\boldsymbol{\iota}}_t$, given by

$$\mathring{\boldsymbol{k}}_t^d = \begin{pmatrix} k_t^{11,d} \\ k_t^{21,d} \\ k_t^{12,d} \\ k_t^{22,d} \end{pmatrix}_{JZ\times 1}, \quad \mathring{\boldsymbol{n}}_t^d = \begin{pmatrix} n_t^{11,d} \\ n_t^{21,d} \\ n_t^{12,d} \\ n_t^{22,d} \end{pmatrix}_{JF\times 1}, \quad \mathring{\boldsymbol{\iota}}_t = \begin{pmatrix} \iota_t^{11} \\ \iota_t^{21} \\ \iota_t^{12} \\ \iota_t^{22} \end{pmatrix}_{ZJ\times 1}.$$

We collect capital good uses $k_t^{jz,d}$, for all dates in the $JZ(T+1) \times 1$ vector $\mathring{\boldsymbol{k}}^d$, factor uses, $n_t^{jf,d}$, for all dates in the $JF(T+1) \times 1$ vector $\mathring{\boldsymbol{n}}^d$, and investment allocations, ι_t^{zj}, for all dates in a $ZJ(T+1) \times 1$ vector $\mathring{\boldsymbol{\iota}}$, given by

$$\mathring{\boldsymbol{k}}^d = \begin{pmatrix} \mathring{\boldsymbol{k}}_0^d \\ \vdots \\ \mathring{\boldsymbol{k}}_t^d \\ \vdots \\ \mathring{\boldsymbol{k}}_T^d \end{pmatrix}_{JZ(T+1)\times 1}, \quad \mathring{\boldsymbol{n}}^d = \begin{pmatrix} \mathring{\boldsymbol{n}}_0^d \\ \vdots \\ \mathring{\boldsymbol{n}}_t^d \\ \vdots \\ \mathring{\boldsymbol{n}}_T^d \end{pmatrix}_{JF(T+1)\times 1}, \quad \mathring{\boldsymbol{\iota}} = \begin{pmatrix} \mathring{\boldsymbol{\iota}}_0 \\ \vdots \\ \mathring{\boldsymbol{\iota}}_t \\ \vdots \\ \mathring{\boldsymbol{\iota}}_T \end{pmatrix}_{ZJ(T+1)\times 1}.$$

Aggregate allocations. We collect aggregate consumption at date t, c_t^j, aggregate investment at date t, ι_t^j, aggregate intermediate use at date t, x_t^j, goods' supply at date t, $y_t^{j,s}$, aggregate goods endowment at date t, $\bar{y}_t^{j,s}$, and aggregate goods supply at date t, y_t^j, in $J \times 1$ vectors $\boldsymbol{c}_t$, $\boldsymbol{\iota}_t$, $\boldsymbol{x}_t$, $\boldsymbol{y}_t^s$, $\bar{\boldsymbol{y}}_t^s$, and $\boldsymbol{y}_t$, given by

$$\boldsymbol{c}_t = \begin{pmatrix} c_t^1 \\ c_t^2 \end{pmatrix}_{J\times 1}, \quad \boldsymbol{\iota}_t = \begin{pmatrix} \iota_t^1 \\ \iota_t^2 \end{pmatrix}_{J\times 1}, \quad \boldsymbol{x}_t = \begin{pmatrix} x_t^1 \\ x_t^2 \end{pmatrix}_{J\times 1},$$

$$\boldsymbol{y}_t^s = \begin{pmatrix} y_t^{1,s} \\ y_t^{2,s} \end{pmatrix}_{J\times 1}, \quad \bar{\boldsymbol{y}}_t^s = \begin{pmatrix} \bar{y}_t^{1,s} \\ \bar{y}_t^{2,s} \end{pmatrix}_{J\times 1}, \quad \boldsymbol{y}_t = \begin{pmatrix} y_t^1 \\ y_t^2 \end{pmatrix}_{J\times 1}.$$

Similarly, we collect aggregate use, $k_t^{z,d}$, supply, $k_t^{z,s}$, aggregate endowment, $\bar{k}_t^{z,s}$, and aggregate supply, k_t^z, of capital goods at date t in $Z \times 1$ vectors $\boldsymbol{k}_t^d$, $\boldsymbol{k}_t^s$, $\bar{\boldsymbol{k}}_t^s$, and $\boldsymbol{k}_t$, given by

$$\boldsymbol{k}_t^d = \begin{pmatrix} k_t^{1,d} \\ k_t^{2,d} \end{pmatrix}_{Z\times 1}, \quad \boldsymbol{k}_t^s = \begin{pmatrix} k_t^{1,s} \\ k_t^{2,s} \end{pmatrix}_{Z\times 1}, \quad \bar{\boldsymbol{k}}_t^s = \begin{pmatrix} \bar{k}_t^{1,s} \\ \bar{k}_t^{2,s} \end{pmatrix}_{Z\times 1}, \quad \boldsymbol{k}_t = \begin{pmatrix} k_t^1 \\ k_t^2 \end{pmatrix}_{Z\times 1}.$$

Analogously, we collect aggregate use, $n_t^{f,d}$, aggregate elastic supply, $n_t^{f,s}$, aggregate endowment, $\bar{n}_t^{f,s}$, and aggregate supply, n_t^f, of factors at date t in $F \times 1$ vectors $\boldsymbol{n}_t^d$, $\boldsymbol{n}_t^s$, $\bar{\boldsymbol{n}}_t^s$, and $\boldsymbol{n}_t$, given by

$$\boldsymbol{n}_t^d = \begin{pmatrix} n_t^{1,d} \\ n_t^{2,d} \end{pmatrix}_{F\times 1}, \quad \boldsymbol{n}_t^s = \begin{pmatrix} n_t^{1,s} \\ n_t^{2,s} \end{pmatrix}_{F\times 1}, \quad \bar{\boldsymbol{n}}_t^s = \begin{pmatrix} \bar{n}_t^{1,s} \\ \bar{n}_t^{2,s} \end{pmatrix}_{F\times 1}, \quad \boldsymbol{n}_t = \begin{pmatrix} n_t^1 \\ n_t^2 \end{pmatrix}_{F\times 1}.$$

We collect aggregate consumption, c_t^j, aggregate investment, ι_t^j, aggregate intermediate use, x_t^j, goods' supply, $y_t^{j,s}$, aggregate goods endowment, $\bar{y}_t^{j,s}$, and aggregate goods supply, y_t^j, for all dates in $J(T+1) \times 1$ vectors $\boldsymbol{c}$, $\boldsymbol{\iota}$, $\boldsymbol{x}$, $\boldsymbol{y}^s$, $\bar{\boldsymbol{y}}^s$, and $\boldsymbol{y}$, given by

$$\boldsymbol{c} = \begin{pmatrix} \boldsymbol{c}_0 \\ \vdots \\ \boldsymbol{c}_t \\ \vdots \\ \boldsymbol{c}_T \end{pmatrix}_{J(T+1)\times 1}, \quad \boldsymbol{\iota} = \begin{pmatrix} \boldsymbol{\iota}_0 \\ \vdots \\ \boldsymbol{\iota}_t \\ \vdots \\ \boldsymbol{\iota}_T \end{pmatrix}_{J(T+1)\times 1}, \quad \boldsymbol{x} = \begin{pmatrix} \boldsymbol{x}_0 \\ \vdots \\ \boldsymbol{x}_t \\ \vdots \\ \boldsymbol{x}_T \end{pmatrix}_{J(T+1)\times 1},$$

$$\boldsymbol{y}^s = \begin{pmatrix} \boldsymbol{y}_0^s \\ \vdots \\ \boldsymbol{y}_t^s \\ \vdots \\ \boldsymbol{y}_T^s \end{pmatrix}_{J(T+1)\times 1}, \quad \bar{\boldsymbol{y}}^s = \begin{pmatrix} \bar{\boldsymbol{y}}_0^s \\ \vdots \\ \bar{\boldsymbol{y}}_t^s \\ \vdots \\ \bar{\boldsymbol{y}}_T^s \end{pmatrix}_{J(T+1)\times 1}, \quad \boldsymbol{y} = \begin{pmatrix} \boldsymbol{y}_0 \\ \vdots \\ \boldsymbol{y}_t \\ \vdots \\ \boldsymbol{y}_T \end{pmatrix}_{J(T+1)\times 1}.$$

Similarly, we collect aggregate use, $k_t^{z,d}$, supply, $k_t^{z,s}$, aggregate endowment, $\bar{k}_t^{z,s}$, and aggregate supply, k_t^z, of capital goods for all dates in $Z(T+1) \times 1$ vectors $\boldsymbol{k}^d$, $\boldsymbol{k}^s$, $\bar{\boldsymbol{k}}^s$, and $\boldsymbol{k}$, given by

$$\boldsymbol{k}^d = \begin{pmatrix} \boldsymbol{k}_0^d \\ \vdots \\ \boldsymbol{k}_t^d \\ \vdots \\ \boldsymbol{k}_T^d \end{pmatrix}_{Z(T+1)\times 1}, \quad \boldsymbol{k}^s = \begin{pmatrix} \boldsymbol{k}_0^s \\ \vdots \\ \boldsymbol{k}_t^s \\ \vdots \\ \boldsymbol{k}_T^s \end{pmatrix}_{Z(T+1)\times 1}, \quad \bar{\boldsymbol{k}}^s = \begin{pmatrix} \bar{\boldsymbol{k}}_0^s \\ \vdots \\ \bar{\boldsymbol{k}}_t^s \\ \vdots \\ \bar{\boldsymbol{k}}_T^s \end{pmatrix}_{Z(T+1)\times 1}, \quad \boldsymbol{k} = \begin{pmatrix} \boldsymbol{k}_0 \\ \vdots \\ \boldsymbol{k}_t \\ \vdots \\ \boldsymbol{k}_T \end{pmatrix}_{Z(T+1)\times 1}.$$

Analogously, we collect aggregate use, $n_t^{f,d}$, aggregate elastic supply, $n_t^{f,s}$, aggregate endowment, $\bar{n}_t^{f,s}$, and aggregate supply, n_t^f, of factors for all dates in $F(T+1) \times 1$ vectors $\boldsymbol{n}^d$, $\boldsymbol{n}^s$, $\bar{\boldsymbol{n}}^s$, and $\boldsymbol{n}$, given by

$$\boldsymbol{n}^d = \begin{pmatrix} \boldsymbol{n}_0^d \\ \vdots \\ \boldsymbol{n}_t^d \\ \vdots \\ \boldsymbol{n}_T^d \end{pmatrix}_{F(T+1)\times 1}, \quad \boldsymbol{n}^s = \begin{pmatrix} \boldsymbol{n}_0^s \\ \vdots \\ \boldsymbol{n}_t^s \\ \vdots \\ \boldsymbol{n}_T^s \end{pmatrix}_{F(T+1)\times 1}, \quad \bar{\boldsymbol{n}}^s = \begin{pmatrix} \bar{\boldsymbol{n}}_0^s \\ \vdots \\ \bar{\boldsymbol{n}}_t^s \\ \vdots \\ \bar{\boldsymbol{n}}_T^s \end{pmatrix}_{F(T+1)\times 1}, \quad \boldsymbol{n} = \begin{pmatrix} \boldsymbol{n}_0 \\ \vdots \\ \boldsymbol{n}_t \\ \vdots \\ \boldsymbol{n}_T \end{pmatrix}_{F(T+1)\times 1}.$$

Allocation shares. We collect factor use shares at date t, $\chi_{t,n}^{jf,d}$, in a $JF \times F$ matrix $\boldsymbol{\chi}_{t,n^d}$, intermediate use shares at date t, $\chi_{t,x}^{j\ell}$, in a $JL \times L$ matrix $\boldsymbol{\chi}_{t,x}$, intermediate-supply shares at date t, $\xi_t^{j\ell}$, in a $JL \times L$ matrix $\boldsymbol{\xi}_t$, capital use shares at date t, $\chi_{t,k}^{jz,d}$, in a $JZ \times Z$ matrix $\boldsymbol{\chi}_{t,k^d}$, and investment use shares at date t, $\chi_{t,\iota}^{zj}$, in a $ZJ \times J$ matrix $\boldsymbol{\chi}_{t,\iota}$, given by

$$\boldsymbol{\chi}_{t,n^d} = \begin{pmatrix} \chi_{t,n}^{11,d} & 0 \\ \chi_{t,n}^{21,d} & 0 \\ 0 & \chi_{t,n}^{12,d} \\ 0 & \chi_{t,n}^{22,d} \end{pmatrix}_{JF\times F}, \quad \boldsymbol{\chi}_{t,x} = \begin{pmatrix} \chi_{t,x}^{11} & 0 \\ \chi_{t,x}^{21} & 0 \\ 0 & \chi_{t,x}^{12} \\ 0 & \chi_{t,x}^{22} \end{pmatrix}_{JL\times L}, \quad \boldsymbol{\xi}_t = \begin{pmatrix} \xi_t^{11} & 0 \\ \xi_t^{21} & 0 \\ 0 & \xi_t^{12} \\ 0 & \xi_t^{22} \end{pmatrix}_{JL\times L},$$

$$\boldsymbol{\chi}_{t,k^d} = \begin{pmatrix} \chi_{t,k}^{11,d} & 0 \\ \chi_{t,k}^{21,d} & 0 \\ 0 & \chi_{t,k}^{12,d} \\ 0 & \chi_{t,k}^{22,d} \end{pmatrix}_{JZ\times Z}, \quad \boldsymbol{\chi}_{t,\iota} = \begin{pmatrix} \chi_{t,\iota}^{11} & 0 \\ \chi_{t,\iota}^{21} & 0 \\ 0 & \chi_{t,\iota}^{12} \\ 0 & \chi_{t,\iota}^{22} \end{pmatrix}_{ZJ\times J}.$$

We collect aggregate consumption, aggregate intermediate input, and aggregate investment shares, $\phi_{t,c}^j$, $\phi_{t,x}^j$, and $\phi_{t,\iota}^j$, at date t in $J \times J$ diagonal matrices, $\boldsymbol{\phi}_{t,c}$, $\boldsymbol{\phi}_{t,x}$, and $\boldsymbol{\phi}_{t,\iota}$, given by

$$\boldsymbol{\phi}_{t,c} = \begin{pmatrix} \phi_{t,c}^1 & 0 \\ 0 & \phi_{t,c}^2 \end{pmatrix}_{J\times J}, \quad \boldsymbol{\phi}_{t,x} = \begin{pmatrix} \phi_{t,x}^1 & 0 \\ 0 & \phi_{t,x}^2 \end{pmatrix}_{J\times J}, \quad \boldsymbol{\phi}_{t,\iota} = \begin{pmatrix} \phi_{t,\iota}^1 & 0 \\ 0 & \phi_{t,\iota}^2 \end{pmatrix}_{J\times J}.$$

We collect factor use shares, $\chi_{t,n}^{jf,d}$, for all dates in a $JF(T+1) \times F(T+1)$ block-diagonal matrix $\boldsymbol{\chi}_{n^d}$, intermediate use shares, $\chi_{t,x}^{j\ell}$, for all dates in a $JL(T+1) \times L(T+1)$ block-diagonal matrix $\boldsymbol{\chi}_x$, intermediate-supply shares, $\xi_t^{j\ell}$, for all dates in a $JL(T+1) \times L(T+1)$ block-diagonal matrix $\boldsymbol{\xi}$, capital use shares, $\chi_{t,k}^{jz,d}$, in a $JZ(T+1) \times Z(T+1)$ block-diagonal matrix $\boldsymbol{\chi}_{k^d}$, and investment use shares, $\chi_{t,\iota}^{zj}$, for all dates in a $ZJ(T+1) \times J(T+1)$ block-diagonal matrix $\boldsymbol{\chi}_\iota$, given by

$$\boldsymbol{\chi}_{n^d} = \operatorname{diag}\left(\boldsymbol{\chi}_{0,n^d}, \ldots, \boldsymbol{\chi}_{t,n^d}, \ldots, \boldsymbol{\chi}_{T,n^d}\right), \quad \boldsymbol{\chi}_x = \operatorname{diag}\left(\boldsymbol{\chi}_{0,x}, \ldots, \boldsymbol{\chi}_{t,x}, \ldots, \boldsymbol{\chi}_{T,x}\right),$$
$$\boldsymbol{\xi} = \operatorname{diag}\left(\boldsymbol{\xi}_0, \ldots, \boldsymbol{\xi}_t, \ldots, \boldsymbol{\xi}_T\right), \quad \boldsymbol{\chi}_{k^d} = \operatorname{diag}\left(\boldsymbol{\chi}_{0,k^d}, \ldots, \boldsymbol{\chi}_{t,k^d}, \ldots, \boldsymbol{\chi}_{T,k^d}\right),$$
$$\boldsymbol{\chi}_\iota = \operatorname{diag}\left(\boldsymbol{\chi}_{0,\iota}, \ldots, \boldsymbol{\chi}_{t,\iota}, \ldots, \boldsymbol{\chi}_{T,\iota}\right).$$

We collect aggregate consumption, aggregate intermediate input, and aggregate investment shares, $\phi_{t,c}^j$, $\phi_{t,x}^j$, and $\phi_{t,\iota}^j$, for all dates in $J(T+1) \times J(T+1)$ block-diagonal matrices, $\boldsymbol{\phi}_c$, $\boldsymbol{\phi}_x$, and $\boldsymbol{\phi}_\iota$, given by

$$\boldsymbol{\phi}_c = \operatorname{diag}\left(\boldsymbol{\phi}_{0,c}, \ldots, \boldsymbol{\phi}_{t,c}, \ldots, \boldsymbol{\phi}_{T,c}\right), \quad \boldsymbol{\phi}_x = \operatorname{diag}\left(\boldsymbol{\phi}_{0,x}, \ldots, \boldsymbol{\phi}_{t,x}, \ldots, \boldsymbol{\phi}_{T,x}\right),$$
$$\boldsymbol{\phi}_\iota = \operatorname{diag}\left(\boldsymbol{\phi}_{0,\iota}, \ldots, \boldsymbol{\phi}_{t,\iota}, \ldots, \boldsymbol{\phi}_{T,\iota}\right).$$

Marginal accumulation products/accumulation technology change. We collect marginal accumulation products of capital at date $t+1$ in a $Z \times JZ$ matrix $\boldsymbol{H}_{t+1,k}$, marginal accumulation products of investment at date $t+1$ in a $Z \times ZJ$ matrix $\boldsymbol{H}_{t+1,\iota}$, and accumulation technology changes at date $t+1$ in a $Z \times 1$ vector $\boldsymbol{H}_{t+1,\theta}$, given by

$$\boldsymbol{H}_{t+1,k} = \begin{pmatrix} \frac{\partial H_{t+1}^1}{\partial k_t^{11,d}} & \frac{\partial H_{t+1}^1}{\partial k_t^{21,d}} & 0 & 0 \\ 0 & 0 & \frac{\partial H_{t+1}^2}{\partial k_t^{12,d}} & \frac{\partial H_{t+1}^2}{\partial k_t^{22,d}} \end{pmatrix}_{Z \times JZ},$$
$$\boldsymbol{H}_{t+1,\iota} = \begin{pmatrix} \frac{\partial H_{t+1}^1}{\partial \iota_t^{11}} & 0 & \frac{\partial H_{t+1}^1}{\partial \iota_t^{12}} & 0 \\ 0 & \frac{\partial H_{t+1}^2}{\partial \iota_t^{21}} & 0 & \frac{\partial H_{t+1}^2}{\partial \iota_t^{22}} \end{pmatrix}_{Z \times ZJ}, \quad \boldsymbol{H}_{t+1,\theta} = \begin{pmatrix} \frac{\partial H_{t+1}^1}{\partial \theta} \\ \frac{\partial H_{t+1}^2}{\partial \theta} \end{pmatrix}_{Z \times 1}.$$

We collect marginal accumulation products of capital for all dates in a $Z(T+1) \times JZ(T+1)$ matrix $\boldsymbol{H}_k$, marginal accumulation products of investment for all dates in a $Z(T+1) \times ZJ(T+1)$ matrix $\boldsymbol{H}_\iota$, and accumulation technology changes for all dates in a $Z(T+1) \times 1$

vector $\boldsymbol{H}_\theta$, given by

$$\boldsymbol{H}_k = \boldsymbol{\mathcal{S}}_r \text{diag}\left(\boldsymbol{H}_{1,k}, \ldots, \boldsymbol{H}_{t,k}, \ldots, \boldsymbol{H}_{T,k}\right) \boldsymbol{\mathcal{S}}_c,$$

$$\boldsymbol{H}_\iota = \boldsymbol{\mathcal{S}}_r \text{diag}\left(\boldsymbol{H}_{1,\iota}, \ldots, \boldsymbol{H}_{t,\iota}, \ldots, \boldsymbol{H}_{T,\iota}\right) \boldsymbol{\mathcal{S}}_c, \quad \boldsymbol{H}_\theta = \begin{pmatrix} \boldsymbol{H}_{0,\theta} \\ \vdots \\ \boldsymbol{H}_{t,\theta} \\ \vdots \\ \boldsymbol{H}_{T,\theta} \end{pmatrix}_{Z(T+1)\times 1},$$

where $\text{diag}\left(\boldsymbol{H}_{1,k}, \ldots, \boldsymbol{H}_{t,k}, \ldots, \boldsymbol{H}_{T,k}\right)$ and $\text{diag}\left(\boldsymbol{H}_{1,\iota}, \ldots, \boldsymbol{H}_{t,\iota}, \ldots, \boldsymbol{H}_{T,\iota}\right)$ denote block-diagonal matrices of dimensions $ZT \times JZT$ and $ZT \times ZJT$, respectively, whereas $\boldsymbol{\mathcal{S}}_r$ and $\boldsymbol{\mathcal{S}}_c$ are selection matrices given by

$$\boldsymbol{\mathcal{S}}_r = \begin{pmatrix} \mathbf{0}_{Z\times ZT} \\ \boldsymbol{I}_{ZT} \end{pmatrix}_{Z(T+1)\times ZT} \quad \text{and} \quad \boldsymbol{\mathcal{S}}_c = \begin{pmatrix} \boldsymbol{I}_{JZT} & \mathbf{0}_{JZT\times JZ} \end{pmatrix}_{JZT\times JZ(T+1)},$$

where $\mathbf{0}_{Z\times ZT}$ and $\mathbf{0}_{JZT\times JZ}$ denote zero matrices of dimensions $Z \times ZT$ and $JZT \times JZ$, respectively, whereas $\boldsymbol{I}_{ZT}$ and $\boldsymbol{I}_{JZT}$ denote identity matrices of dimensions ZT and JZT, respectively.

Marginal products/production technology change. We collect marginal products of capital at date t in a $J \times JZ$ matrix $\boldsymbol{G}_{t,k}$, marginal products of factors at date t in a $J \times JF$ matrix $\boldsymbol{G}_{t,n}$, marginal products of intermediates at date t in a $J \times JL$ matrix $\boldsymbol{G}_{t,x}$, and production technology changes at date t in a $J \times 1$ vector $\boldsymbol{G}_{t,\theta}$, given by

$$\boldsymbol{G}_{t,k} = \begin{pmatrix} \frac{\partial G_t^1}{\partial k_t^{11,d}} & 0 & \frac{\partial G_t^1}{\partial k_t^{12,d}} & 0 \\ 0 & \frac{\partial G_t^2}{\partial k_t^{21,d}} & 0 & \frac{\partial G_t^2}{\partial k_t^{22,d}} \end{pmatrix}_{J\times JZ}, \quad \boldsymbol{G}_{t,x} = \begin{pmatrix} \frac{\partial G_t^1}{\partial x_t^{11}} & 0 & \frac{\partial G_t^1}{\partial x_t^{12}} & 0 \\ 0 & \frac{\partial G_t^2}{\partial x_t^{21}} & 0 & \frac{\partial G_t^2}{\partial x_t^{22}} \end{pmatrix}_{J\times JL},$$

$$\boldsymbol{G}_{t,n} = \begin{pmatrix} \frac{\partial G_t^1}{\partial n_t^{11,d}} & 0 & \frac{\partial G_t^1}{\partial n_t^{12,d}} & 0 \\ 0 & \frac{\partial G_t^2}{\partial n_t^{21,d}} & 0 & \frac{\partial G_t^2}{\partial n_t^{22,d}} \end{pmatrix}_{J\times JF}, \quad \boldsymbol{G}_{t,\theta} = \begin{pmatrix} \frac{\partial G_t^1}{\partial \theta} \\ \frac{\partial G_t^2}{\partial \theta} \end{pmatrix}_{J\times 1}.$$

We collect marginal products of capital for all dates in a $J(T+1) \times JZ(T+1)$ block-diagonal matrix $\boldsymbol{G}_k$, marginal products of factors for all dates in a $J(T+1) \times JF(T+1)$ block-diagonal matrix $\boldsymbol{G}_n$, marginal products of intermediates for all dates in a $J(T+1) \times JL(T+1)$ block-diagonal matrix $\boldsymbol{G}_x$, and production technology changes for all dates in a $J(T+1) \times 1$ vector

$\boldsymbol{G}_{\theta}$, given by

$$\boldsymbol{G}_{k}=\operatorname{diag}\left(\boldsymbol{G}_{0,k},\ldots,\boldsymbol{G}_{t,k},\ldots,\boldsymbol{G}_{T,k}\right),\quad \boldsymbol{G}_{x}=\operatorname{diag}\left(\boldsymbol{G}_{0,x},\ldots,\boldsymbol{G}_{t,x},\ldots,\boldsymbol{G}_{T,x}\right),$$

$$\boldsymbol{G}_{n}=\operatorname{diag}\left(\boldsymbol{G}_{0,n},\ldots,\boldsymbol{G}_{t,n},\ldots,\boldsymbol{G}_{T,n}\right),\quad \boldsymbol{G}_{\theta}=\begin{pmatrix}\boldsymbol{G}_{0,\theta}\\ \vdots\\ \boldsymbol{G}_{t,\theta}\\ \vdots\\ \boldsymbol{G}_{T,\theta}\end{pmatrix}_{J(T+1)\times 1}.$$

Aggregate marginal rates of substitution. We collect aggregate marginal rates of substitution at date t in $1\times J$ and $1\times F$ vectors $\boldsymbol{AMRS}_{t,c}$ and $\boldsymbol{AMRS}_{t,n}$, given by

$$\boldsymbol{AMRS}_{t,c}=\begin{pmatrix} AMRS_{t,c}^{1} & AMRS_{t,c}^{2}\end{pmatrix}_{1\times J}\quad\text{and}\quad \boldsymbol{AMRS}_{t,n}=\begin{pmatrix} AMRS_{t,n}^{1} & AMRS_{t,n}^{2}\end{pmatrix}_{1\times F}.$$

We collect aggregate marginal rates of substitution for all dates in $(T+1)\times J\,(T+1)$ and $(T+1)\times F\,(T+1)$ block-diagonal matrices $\boldsymbol{AMRS}_{c}$ and $\boldsymbol{AMRS}_{n}$, given by

$$\begin{aligned}\boldsymbol{AMRS}_{c}&=\operatorname{diag}\left(\boldsymbol{AMRS}_{0,c},\ldots,\boldsymbol{AMRS}_{t,c},\ldots,\boldsymbol{AMRS}_{T,c}\right),\\ \boldsymbol{AMRS}_{n}&=\operatorname{diag}\left(\boldsymbol{AMRS}_{0,n},\ldots,\boldsymbol{AMRS}_{t,n},\ldots,\boldsymbol{AMRS}_{T,n}\right).\end{aligned}$$

Marginal social values. We collect marginal social values at date t in $1\times J$ and $1\times Z$ vectors $\boldsymbol{MSV}_{t,y}$ and $\boldsymbol{MSV}_{t,k}$, given by

$$\boldsymbol{MSV}_{t,y}=\begin{pmatrix} MSV_{t,y}^{1} & MSV_{t,y}^{2}\end{pmatrix}_{1\times J}\quad\text{and}\quad \boldsymbol{MSV}_{t,k}=\begin{pmatrix} MSV_{t,k}^{1} & MSV_{t,k}^{2}\end{pmatrix}_{1\times Z}.$$

We collect marginal social values for all dates in $1\times J\,(T+1)$ and $1\times Z\,(T+1)$ vectors $\boldsymbol{MSV}_{y}$ and $\boldsymbol{MSV}_{k}$, given by

$$\begin{aligned}\boldsymbol{MSV}_{y}&=\left(\boldsymbol{MSV}_{0,y},\ldots,\boldsymbol{MSV}_{t,y},\ldots,\boldsymbol{MSV}_{T,y}\right)_{1\times J(T+1)},\\ \boldsymbol{MSV}_{k}&=\left(\boldsymbol{MSV}_{0,k},\ldots,\boldsymbol{MSV}_{t,k},\ldots,\boldsymbol{MSV}_{T,k}\right)_{1\times Z(T+1)},\end{aligned}$$

where $\boldsymbol{MSV}_{y}=\boldsymbol{\omega}\boldsymbol{AMRS}_{c}\boldsymbol{\phi}_{c}\boldsymbol{\Psi}_{y}$ and $\boldsymbol{MSV}_{k}=\boldsymbol{\omega}\boldsymbol{AMRS}_{c}\boldsymbol{\phi}_{c}\boldsymbol{\Psi}_{k}$.

Marginal welfare products. We collect marginal welfare products for all dates in $1\times JZ\,(T+1)$, $1\times ZJ\,(T+1)$, $1\times JL\,(T+1)$, and $1\times JF\,(T+1)$ matrices $\boldsymbol{MWP}_{k}$, $\boldsymbol{MWP}_{\iota}$, $\boldsymbol{MWP}_{x}$, and $\boldsymbol{MWP}_{n}$, given by

$$\begin{aligned}\boldsymbol{MWP}_{k}&=\boldsymbol{MSV}_{y}\boldsymbol{G}_{k}+\boldsymbol{MSV}_{k}\boldsymbol{H}_{k}, &\qquad \boldsymbol{MWP}_{\iota}&=\boldsymbol{MSV}_{k}\boldsymbol{H}_{\iota},\\ \boldsymbol{MWP}_{x}&=\boldsymbol{MSV}_{y}\boldsymbol{G}_{x}, & \boldsymbol{MWP}_{n}&=\boldsymbol{MSV}_{y}\boldsymbol{G}_{n}.\end{aligned}$$

We collect aggregate marginal welfare products for all dates in $1\times Z\,(T+1)$, $1\times J\,(T+1)$, $1\times L\,(T+1)$, and $1\times F\,(T+1)$ matrices $\boldsymbol{AMWP}_{k}$, $\boldsymbol{AMWP}_{\iota}$, $\boldsymbol{AMWP}_{x}$, and $\boldsymbol{AMWP}_{n}$, given by

$$\boldsymbol{AMWP}_k = \boldsymbol{MWP}_k \boldsymbol{\chi}_{k^d}, \qquad \boldsymbol{AMWP}_\iota = \boldsymbol{MWP}_\iota \boldsymbol{\chi}_\iota,$$
$$\boldsymbol{AMWP}_x = \boldsymbol{MWP}_x \boldsymbol{\chi}_x, \qquad \boldsymbol{AMWP}_n = \boldsymbol{MWP}_n \boldsymbol{\chi}_{n^d}.$$

Prices. In competitive economies, we collect goods' prices p_t^j at date t in the $1 \times J$ vector $\boldsymbol{p}_t$, capital goods' prices q_t^z at date t in the $1 \times Z$ vector $\boldsymbol{q}_t$, and capital rental rates r_t^z at date t in the $1 \times Z$ vector $\boldsymbol{r}_t$, given by

$$\boldsymbol{p}_t = \begin{pmatrix} p_t^1 & p_t^2 \end{pmatrix}_{1\times J}, \quad \boldsymbol{q}_t = \begin{pmatrix} q_t^1 & q_t^2 \end{pmatrix}_{1\times Z}, \quad \boldsymbol{r}_t = \begin{pmatrix} r_t^1 & r_t^2 \end{pmatrix}_{1\times Z}.$$

We collect goods' prices p_t^j for all dates in the $(T+1) \times J(T+1)$ block-diagonal matrix $\boldsymbol{p}$, capital goods' prices q_t^z for all dates in the $(T+1) \times Z(T+1)$ block-diagonal matrix $\boldsymbol{q}$, and capital rental rates r_t^z for all dates in the $(T+1) \times Z(T+1)$ block-diagonal matrix $\boldsymbol{r}$, given by

$$\boldsymbol{p} = \text{diag}(\boldsymbol{p}_0, \ldots, \boldsymbol{p}_t, \ldots, \boldsymbol{p}_T), \quad \boldsymbol{q} = \text{diag}(\boldsymbol{q}_0, \ldots, \boldsymbol{q}_t, \ldots, \boldsymbol{q}_T), \quad \boldsymbol{r} = \text{diag}(\boldsymbol{r}_0, \ldots, \boldsymbol{r}_t, \ldots, \boldsymbol{r}_T).$$

B Extensions: Generalized Environments

In this section, we extend our results to more general environments. Section B.1 extends the production efficiency decomposition to economies with intermediate inputs, multiple factors, and endowments of goods, factors, and capital. We work directly with production efficiency, $\Xi^{AE,P}$, which makes the results directly applicable to models with a single (representative) individual. Section B.2 shows how welfare assessments in economies with heterogeneous individuals recover $\Xi^{AE,P}$ as one of their components.

B.1 Intermediate Inputs, Multiple Factors, and Endowments

B.1.1 Preferences, Technologies, and Resource Constraints

Here, we extend the production efficiency decomposition to economies with multiple factors, intermediate inputs, and endowments of goods, factors, and capital. We consider an economy with a single individual, which allows us to abstract from redistributional considerations and to focus on characterizing (Kaldor–Hicks) efficiency gains associated with the economy's production side.[20] Throughout this subsection, we work directly with production efficiency, $\Xi^{AE,P}$, which coincides with the welfare assessment of a perturbation in economies with a single individual — Lemma 1. Section B.2 shows that $\Xi^{AE,P}$ remains one of the components of welfare assessments in economies with heterogeneous individuals.

[20]Kaldor–Hicks efficiency corresponds to the unweighted sum of individual willingness-to-pay for the perturbation in units of the lifetime welfare numeraire. Hence, perturbations in which $\Xi^{AE,P} > 0$ can be turned into Pareto improvements if transfers are feasible and costless.

We extend the baseline environment by assuming that there are $F \geq 0$ factors, indexed by $f \in \mathcal{F} = \{1, \ldots, F\}$. In that case, preferences are given by

$$V = \sum_t \beta^t u_t \left(\left\{ c_t^j \right\}_{j \in \mathcal{J}}, \left\{ n_t^{f,s} \right\}_{f \in \mathcal{F}} \right),$$

where $n_t^{f,s}$ denotes the amount of factor f supplied at date t.

Goods are produced using technologies that take goods, capital goods, and factors as inputs. The production technology for good j at date t is given by

$$y_t^{j,s} = G_t^j \left(\left\{ x_t^{j\ell} \right\}_{\ell \in \mathcal{J}}, \left\{ k_t^{jz,d} \right\}_{z \in \mathcal{Z}}, \left\{ n_t^{jf,d} \right\}_{f \in \mathcal{F}}; \theta \right), \tag{OA1}$$

where $y_t^{j,s}$ denotes the supply of good j produced at date t, $x_t^{j\ell}$ denotes the amount of good ℓ used in the production of good j at date t, and $n_t^{jf,d}$ denotes the amount of factor f used in the production of good j at date t. The accumulation technologies are the same as in the body of the paper.

The economy is also endowed with exogenous amounts of goods, factors, and capital, denoted by $\bar{y}_t^{j,s}(\theta)$, $\bar{n}_t^{f,s}(\theta)$, and $\bar{k}_t^{z,s}(\theta)$, which are potentially impacted by the perturbation. The resource constraints are given by

$$y_t^j = c_t^j + \iota_t^j + x_t^j, \quad k_t^{z,d} = k_t^{z,s} + \bar{k}_t^{z,s}(\theta), \quad \text{and} \quad n_t^{f,d} = n_t^{f,s} + \bar{n}_t^{f,s}(\theta),$$

where $y_t^j = y_t^{j,s} + \bar{y}_t^{j,s}(\theta)$ denotes the aggregate amount of good j available at date t, $x_t^j = \sum_\ell x_t^{\ell j}$, and $n_t^{f,d} = \sum_j n_t^{jf,d}$. When $\bar{y}_t^{j,s} = \bar{n}_t^{f,s} = \bar{k}_t^{z,s} = 0$ for all dates, this environment reduces to the one in the body of the paper, augmented with intermediate inputs and factors.

B.1.2 Production Efficiency

The normalized date-t welfare gains — Equation (6) — now account for changes in factor supply, so production efficiency is given by

$$\underbrace{\Xi^{AE,P}}_{\substack{\text{Production}\\ \text{Efficiency}}} = \sum_t \omega_t \Xi_t^{AE,P}, \quad \text{where} \quad \Xi_t^{AE,P} = \frac{dV_t^\lambda}{d\theta} = \sum_j AMRS_{t,c}^j \frac{dc_t^j}{d\theta} - \sum_f AMRS_{t,n}^f \frac{dn_t^{f,s}}{d\theta},$$

where $AMRS_{t,n}^f = -\frac{\beta^t \frac{\partial u_t}{\partial n_t^{f,s}}}{\lambda_t}$ denotes the aggregate marginal rate of substitution between the supply of factor f and the date-t welfare numeraire, which is positive when supplying factors is undesirable. Production efficiency is the object we decompose in the remainder of this subsection.

In addition to the shares defined in the main text, we define (i) *intermediate share*, $\phi_{t,x}^j = x_t^j / y_t^j$, which represents the share of good j devoted to intermediate use in production at date t, (ii) *intermediate-use share*, $\chi_{t,x}^{j\ell} = x_t^{j\ell} / x_t^\ell$, which represents the share of good ℓ's aggregate intermediate input used in the production of good j at date t, and (iii) *intermediate-supply share*,

$\xi_t^{j\ell} = \chi_{t,x}^{j\ell}\phi_{t,x}^{\ell} = x_t^{j\ell}/y_t^{\ell}$, as the product of the two.

Hence, changes in intermediate use are given by

$$\frac{dx_t^{j\ell}}{d\theta} = \frac{d\xi_t^{j\ell}}{d\theta}y_t^{\ell} + \xi_t^{j\ell}\frac{dy_t^{\ell}}{d\theta}, \quad \text{where} \quad \frac{d\xi_t^{j\ell}}{d\theta} = \frac{d\chi_{t,x}^{j\ell}}{d\theta}\phi_{t,x}^{\ell} + \chi_{t,x}^{j\ell}\frac{d\phi_{t,x}^{\ell}}{d\theta}. \tag{OA2}$$

We also define the factor use share, $\chi_{t,n}^{jf,d} = n_t^{jf,d}/n_t^{f,d}$, which represents good j's factor f use as a share of the aggregate use of factor f at date t, so changes in factor use are given by

$$\frac{dn_t^{jf,d}}{d\theta} = \frac{d\chi_{t,n}^{jf,d}}{d\theta}n_t^{f,d} + \chi_{t,n}^{jf,d}\frac{dn_t^{f,d}}{d\theta},$$

where factor market clearing implies that $\frac{dn_t^{f,d}}{d\theta} = \frac{dn_t^{f,s}}{d\theta} + \frac{d\bar{n}_t^{f,s}}{d\theta}$. Finally, changes in the goods, factor, and capital endowments, $\frac{d\bar{y}_t^{j,s}}{d\theta}$, $\frac{d\bar{n}_t^{f,s}}{d\theta}$, and $\frac{d\bar{k}_t^{z,s}}{d\theta}$, are exogenously determined by the perturbation.

B.1.3 Network Propagation

Lemma 4 generalizes Lemma 3 to account for intermediate inputs, factors, and endowments.

Lemma 4. *(Goods Inverse Matrices) Changes in the aggregate supply of good j, $\frac{dy_t^j}{d\theta}$, can be expressed in terms of changes in production technology $\frac{\partial G_t^j}{\partial\theta}$, changes in intermediate use shares $\frac{d\chi_{t,x}^{j\ell}}{d\theta}$, changes in intermediate shares $\frac{d\phi_{t,x}^j}{d\theta}$, changes in capital use shares $\frac{d\chi_{t,k}^{jz,d}}{d\theta}$, changes in factor use $\frac{dn_t^{jf,d}}{d\theta}$, changes in accumulation technology $\frac{\partial H_{t+1}^z}{\partial\theta}$, changes in investment use shares $\frac{d\chi_{t,\iota}^{zj}}{d\theta}$, changes in investment shares $\frac{d\phi_{t,\iota}^j}{d\theta}$, and changes in endowments $\frac{d\bar{y}_t^{j,s}}{d\theta}$, $\frac{d\bar{n}_t^{f,s}}{d\theta}$, and $\frac{d\bar{k}_t^{z,s}}{d\theta}$, as*

$$\begin{aligned} \frac{d\boldsymbol{y}}{d\theta} = {} & \underbrace{\boldsymbol{\Psi}_y}_{\substack{\textit{Goods}\\\textit{Propagation}}} \underbrace{\left(\boldsymbol{G}_\theta + \boldsymbol{G}_x\frac{d\boldsymbol{\xi}}{d\theta}\boldsymbol{y} + \boldsymbol{G}_k\frac{d\boldsymbol{\chi}_{k^d}}{d\theta}\boldsymbol{k}^d + \boldsymbol{G}_n\frac{d\mathring{\boldsymbol{n}}^d}{d\theta} + \boldsymbol{G}_k\boldsymbol{\chi}_{k^d}\frac{d\bar{\boldsymbol{k}}^s}{d\theta} + \frac{d\bar{\boldsymbol{y}}^s}{d\theta}\right)}_{\substack{\textit{Goods}\\\textit{Impulse}}} \\ & + \underbrace{\boldsymbol{\Psi}_k}_{\substack{\textit{Goods-Capital}\\\textit{Propagation}}} \underbrace{\left(\boldsymbol{H}_\theta + \boldsymbol{H}_k\frac{d\boldsymbol{\chi}_{k^d}}{d\theta}\boldsymbol{k}^d + \boldsymbol{H}_\iota\frac{d\boldsymbol{\chi}_\iota}{d\theta}\boldsymbol{\iota} + \boldsymbol{H}_\iota\boldsymbol{\chi}_\iota\frac{d\boldsymbol{\phi}_\iota}{d\theta}\boldsymbol{y} + \boldsymbol{H}_k\boldsymbol{\chi}_{k^d}\frac{d\bar{\boldsymbol{k}}^s}{d\theta}\right)}_{\substack{\textit{Goods-Capital}\\\textit{Impulse}}}, \end{aligned} \tag{OA3}$$

where $\frac{d\boldsymbol{y}}{d\theta}$ denotes the $J(T+1)\times 1$ vector of $\frac{dy_t^j}{d\theta}$, $\frac{d\bar{\boldsymbol{y}}^s}{d\theta}$ and $\frac{d\bar{\boldsymbol{k}}^s}{d\theta}$ denote the $J(T+1)\times 1$ and $Z(T+1)\times 1$ vectors of changes in goods and capital endowments, $\boldsymbol{\Psi}_k = \boldsymbol{\Psi}_y\boldsymbol{G}_k\boldsymbol{\chi}_{k^d}\boldsymbol{\mathcal{A}}_k$ defines the $J(T+1)\times Z(T+1)$ goods-capital inverse matrix, and $\boldsymbol{\Psi}_y = \left(\boldsymbol{I}_{J(T+1)} - \boldsymbol{G}_k\boldsymbol{\chi}_{k^d}\boldsymbol{\mathcal{A}}_k\boldsymbol{H}_\iota\boldsymbol{\chi}_\iota\boldsymbol{\phi}_\iota - \boldsymbol{G}_x\boldsymbol{\xi}\right)^{-1}$ defines the $J(T+1)\times J(T+1)$ goods inverse matrix. The remaining matrices are defined in Appendix A.

We highlight several features of Lemma 4. First, there are two "goods impulse" terms in addition to the ones in the main text due to the existence of intermediate inputs and factors. A perturbation that changes intermediate-supply shares at date t by $\frac{d\xi_t^{j\ell}}{d\theta}$ mechanically increases the amount of good ℓ devoted to the production of good j in proportion to y_t^ℓ, which in turn increases the supply of good j at date t by $\frac{\partial G_t^j}{\partial x_t^{j\ell}}$. Similarly, a perturbation that changes the use of factor f in the production of good j at date t by $\frac{dn_t^{jf,d}}{d\theta}$ mechanically increases the supply of good j at date t by $\frac{\partial G_t^j}{\partial n_t^{jf,d}}$.

Second, changes in endowments act as additional impulses. Changes in goods endowments, $\frac{d\bar{\boldsymbol{y}}^s}{d\theta}$, are direct goods impulses. Changes in capital endowments, $\frac{d\bar{\boldsymbol{k}}^s}{d\theta}$, which are allocated across uses in proportion to the capital use shares $\boldsymbol{\chi}_{k^d}$, increase contemporaneous output through $\boldsymbol{G}_k$ — a goods impulse — and future capital stocks through $\boldsymbol{H}_k$ — a goods-capital impulse. Changes in factor endowments enter through changes in factor use, since $\frac{dn_t^{f,d}}{d\theta} = \frac{dn_t^{f,s}}{d\theta} + \frac{d\bar{n}_t^{f,s}}{d\theta}$.

Third, first-round changes in the aggregate supply of goods induce further changes in the level of intermediate inputs, which in turn induce further changes in the supply of goods in addition to the goods' knock-on effects due to investment previously explained in the main text. Hence, we have static propagation due to input-output linkages emerging from intermediate inputs, and dynamic propagation due to capital accumulation, and both are captured by the goods inverse matrix, $\boldsymbol{\Psi}_y$. In the main text, we only focus on dynamic propagation.

Finally, the main mechanisms behind the goods-capital inverse matrix, $\boldsymbol{\Psi}_k$, remain identical.

B.1.4 Welfare-Relevant Statistics

The marginal social values are identical to the ones in Definition 1. As for the marginal welfare products, we define them for factors and intermediate inputs in addition to the ones previously defined in Definitions 2 and 3.

5. *(Marginal Welfare Product) The marginal welfare products of input ℓ and factor f for technology j at date t are given by*

$$MWP_{t,x}^{j\ell} = MSV_{t,y}^j \frac{\partial G_t^j}{\partial x_t^{j\ell}} \quad and \quad MWP_{t,n}^{jf} = MSV_{t,y}^j \frac{\partial G_t^j}{\partial n_t^{jf,d}}.$$

Marginal welfare products capture the efficiency gain associated with using a factor or an intermediate input. Marginal increases in $x_t^{j\ell}$ or $n_t^{jf,d}$ mechanically increase the supply of good j at date t by their corresponding technological marginal products, $\frac{\partial G_t^j}{\partial x_t^{j\ell}}$ and $\frac{\partial G_t^j}{\partial n_t^{jf,d}}$. Hence, the marginal welfare product of intermediate inputs and factors are given by the product of the corresponding marginal product and the marginal social value of goods.

6. *(Aggregate Marginal Welfare Product) The aggregate marginal welfare products of input j and factor f at date t are given by*

$$AMWP_{t,x}^j = \sum_\ell \chi_{t,x}^{\ell j} MWP_{t,x}^{\ell j} \quad and \quad AMWP_{t,n}^f = \sum_j \chi_{t,n}^{jf,d} MWP_{t,n}^{jf}.$$

The aggregate marginal welfare product of intermediate inputs captures the efficiency gains associated with increasing the amount of aggregate intermediate use of good j in proportion to the intermediate use shares. The aggregate marginal welfare product of factors captures the efficiency gains associated with increasing the use of factor f in proportion to the factor use shares.

B.1.5 Production Efficiency Decomposition

Theorem 3. *(Production Efficiency Decomposition) Production efficiency, $\Xi^{AE,P}$, can be decomposed into (i) cross-sectional intermediate input efficiency, (ii) aggregate intermediate input efficiency, (iii) cross-sectional investment efficiency, (iv) aggregate investment efficiency, (v) cross-sectional factor use efficiency, (vi) aggregate factor efficiency, (vii) cross-sectional capital use efficiency, (viii) accumulation technology change, (ix) production technology change, (x) good endowment growth, (xi) factor endowment growth, and (xii) capital endowment growth, as*

$$
\begin{aligned}
\Xi^{AE,P} = {} & \overbrace{\underbrace{\sum_t \sum_\ell \mathbb{C}ov_j^{\Sigma}\left[MWP_{t,x}^{j\ell}, \frac{d\chi_{t,x}^{j\ell}}{d\theta}\right] x_t^\ell}_{\substack{\text{Cross-Sectional} \\ \text{Intermediate Input Efficiency}}} + \underbrace{\sum_t \sum_\ell \left(AMWP_{t,x}^\ell - \omega_t AMRS_{t,c}^\ell\right) \frac{d\phi_{t,x}^\ell}{d\theta} y_t^\ell}_{\substack{\text{Aggregate} \\ \text{Intermediate Input Efficiency}}}}^{\text{Intermediate Input Efficiency}} \\
& + \overbrace{\underbrace{\sum_t \sum_j \mathbb{C}ov_z^{\Sigma}\left[MWP_{t,\iota}^{zj}, \frac{d\chi_{t,\iota}^{zj}}{d\theta}\right] \iota_t^j}_{\substack{\text{Cross-Sectional} \\ \text{Investment Efficiency}}} + \underbrace{\sum_t \sum_j \left(AMWP_{t,\iota}^j - \omega_t AMRS_{t,c}^j\right) \frac{d\phi_{t,\iota}^j}{d\theta} y_t^j}_{\substack{\text{Aggregate} \\ \text{Investment Efficiency}}}}^{\text{Investment Efficiency}} \\
& + \overbrace{\underbrace{\sum_t \sum_f \mathbb{C}ov_j^{\Sigma}\left[MWP_{t,n}^{jf}, \frac{d\chi_{t,n}^{jf,d}}{d\theta}\right] n_t^{f,d}}_{\substack{\text{Cross-Sectional} \\ \text{Factor Use Efficiency}}} + \underbrace{\sum_t \sum_f \left(AMWP_{t,n}^f - \omega_t AMRS_{t,n}^f\right) \frac{dn_t^{f,s}}{d\theta}}_{\substack{\text{Aggregate} \\ \text{Factor Efficiency}}}}^{\text{Factor Efficiency}} \\
& + \overbrace{\underbrace{\sum_t \sum_z \mathbb{C}ov_j^{\Sigma}\left[MWP_{t,k}^{jz}, \frac{d\chi_{t,k}^{jz,d}}{d\theta}\right] k_t^{z,d}}_{\substack{\text{Cross-Sectional} \\ \text{Capital Use Efficiency}}}}^{\text{Capital Efficiency}} + \underbrace{\sum_t \sum_z MSV_{t,k}^z \frac{\partial H_t^z}{\partial \theta}}_{\substack{\text{Accumulation} \\ \text{Technology Change}}} + \underbrace{\sum_t \sum_j MSV_{t,y}^j \frac{\partial G_t^j}{\partial \theta}}_{\substack{\text{Production} \\ \text{Technology Change}}} \\
& + \overbrace{\underbrace{\sum_t \sum_j MSV_{t,y}^j \frac{d\bar{y}_t^{j,s}}{d\theta}}_{\substack{\text{Good} \\ \text{Endowment Growth}}} + \underbrace{\sum_t \sum_f AMWP_{t,n}^f \frac{d\bar{n}_t^{f,s}}{d\theta}}_{\substack{\text{Factor} \\ \text{Endowment Growth}}} + \underbrace{\sum_t \sum_z AMWP_{t,k}^z \frac{d\bar{k}_t^{z,s}}{d\theta}}_{\substack{\text{Capital} \\ \text{Endowment Growth}}}}^{\text{Endowment Growth}} .
\end{aligned}
$$

The new cross-sectional components correspond to covariances across uses, measuring gains from reallocating intermediate inputs or factors from low to high marginal welfare product uses, for given levels of aggregate intermediate use or factor use.

The endowment growth components value changes in endowments at the corresponding welfare-relevant statistics: changes in goods endowments are valued at the marginal social value of goods, changes in factor endowments at the aggregate marginal welfare product of factors, and changes in capital endowments at the aggregate marginal welfare product of capital. The capital valuation combines the contemporaneous production gains and the future accumulation gains of additional capital, since $MWP^{jz}_{t,k} = MSV^{j}_{t,y} \frac{\partial G^j_t}{\partial k^{jz,d}_t} + MSV^{z}_{t+1,k} \frac{\partial H^z_{t+1}}{\partial k^{jz,d}_t}$.

B.2 Heterogeneous Individuals

Here, we show how to go from welfare assessments in economies with heterogeneous individuals to production efficiency, $\Xi^{AE,P}$, the object decomposed in Section B.1. We consider a perfect foresight economy populated by a finite number $I \geq 1$ of individuals, indexed by $i \in \mathcal{I} = \{1, \ldots, I\}$.

An individual i derives utility from consuming goods and (dis)utility from supplying factors, with a lifetime utility representation given by

$$V^i = \sum_t \left(\beta^i\right)^t u^i_t \left(\left\{ c^{ij}_t \right\}_{j \in \mathcal{J}}, \left\{ n^{if,s}_t \right\}_{f \in \mathcal{F}} \right),$$

where $\beta^i \in [0,1)$ denotes individual i's discount factor, and $u^i_t(\cdot)$ corresponds to individual i's instantaneous utility, c^{ij}_t denotes individual i's final consumption of good j at date t, and $n^{if,s}_t$ denotes individual i's supply of factor f at date t.

We consider welfare assessments for welfarist planners, that is, planners with a social welfare function $\mathcal{W}(\cdot)$ given by

$$W = \mathcal{W}\left(V^1, \ldots, V^i, \ldots, V^I\right).$$

A welfare assessment can be expressed as

$$\frac{dW}{d\theta} = \sum_i \frac{\partial \mathcal{W}}{\partial V^i} \frac{dV^i}{d\theta} = \sum_i \frac{\partial \mathcal{W}}{\partial V^i} \lambda^i \frac{dV^{i|\lambda}}{d\theta},$$

where the normalized individual lifetime welfare gains, $\frac{dV^{i|\lambda}}{d\theta}$, are given by

$$\frac{dV^{i|\lambda}}{d\theta} = \frac{\frac{dV^i}{d\theta}}{\lambda^i} = \sum_t \frac{\lambda^i_t}{\lambda^i} \frac{dV^{i|\lambda}_t}{d\theta}, \quad \text{where} \quad \frac{dV^{i|\lambda}_t}{d\theta} = \sum_j \frac{\left(\beta^i\right)^t \frac{\partial u^i_t}{\partial c^{ij}_t}}{\lambda^i_t} \frac{dc^{ij}_t}{d\theta} + \sum_f \frac{\left(\beta^i\right)^t \frac{\partial u^i_t}{\partial n^{if,s}_t}}{\lambda^i_t} \frac{dn^{if,s}_t}{d\theta},$$

where λ^i is a normalizing factor that allows us to express individual welfare gains or losses in units of a common lifetime welfare numeraire and λ^i_t is a date-t normalizing factor that allows us to express individual welfare gains or losses in units of a common date welfare numeraire.

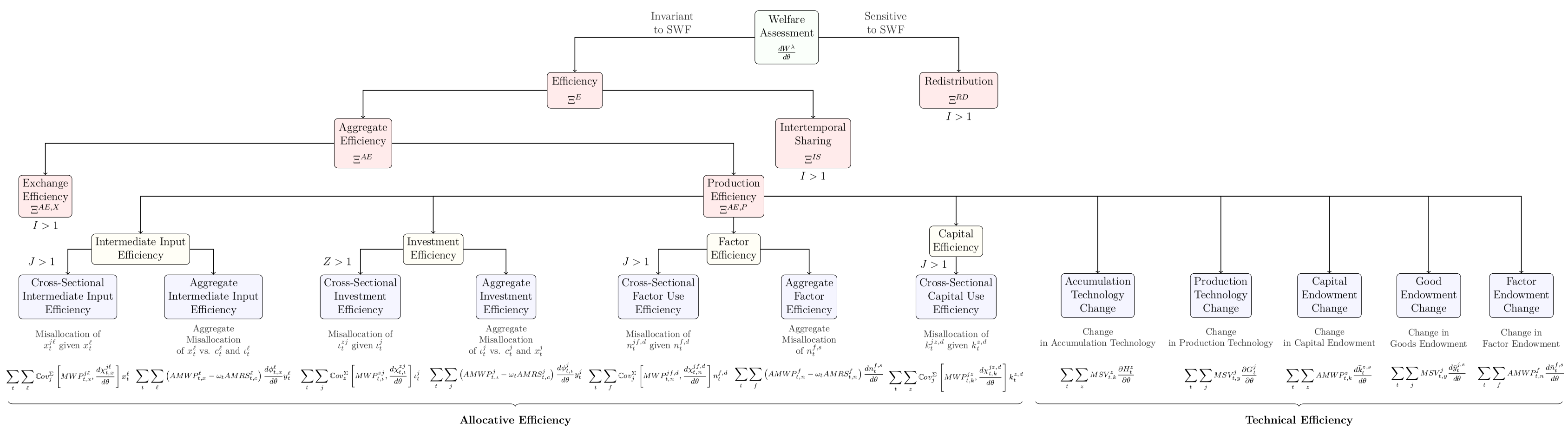


Figure OA1: Welfare Decomposition, with Intermediate Inputs, Factors, and Endowments

Note: This diagram illustrates the production efficiency decomposition of Theorem 3, which generalizes Theorem 1 in the main text (Figure 1) to economies with intermediate inputs, multiple factors, and endowments, supplementing it with intermediate input efficiency, factor efficiency, and endowment growth.

These normalizing factors must be strictly positive for all individuals.

Lemma 5 derives Dávila and Schaab (2025)'s welfare decomposition. Afterwards, we decompose aggregate efficiency as in Dávila and Schaab (2023) into exchange and production efficiency.

Lemma 5. *(Welfare Decomposition) A normalized welfare assessment for a welfarist planner can be decomposed into (i) aggregate efficiency, (ii) intertemporal-sharing, and (iii) redistribution, as*

$$\underbrace{\frac{dW^{\lambda}}{d\theta}}_{\substack{\text{Welfare}\\ \text{Assessment}}} = \overbrace{\underbrace{\sum_t \omega_t \frac{dV_t^{\lambda}}{d\theta}}_{\Xi^{AE}\ \text{(Aggregate Efficiency)}} + \underbrace{\sum_t \omega_t \mathbb{C}ov_i^{\Sigma}\left[\frac{\omega_t^i}{\omega_t}, \frac{dV_t^{i|\lambda}}{d\theta}\right]}_{\Xi^{IS}\ \text{(Intertemporal-Sharing)}}}^{\Xi^{E}\ \text{(Efficiency)}} + \underbrace{\mathbb{C}ov_i^{\Sigma}\left[\omega^i, \frac{dV^{i|\lambda}}{d\theta}\right]}_{\Xi^{RD}\ \text{(Redistribution)}},$$

where the normalized individual weights, ω^i, the normalized dynamic weights, ω_t^i, and the average of the normalized dynamic weights, ω_t, are given by

$$\omega^i = \frac{\frac{\partial \mathcal{W}}{\partial V^i}\lambda^i}{\frac{1}{I}\sum_i \frac{\partial \mathcal{W}}{\partial V^i}\lambda^i} \qquad \text{(Normalized Individual Weight)}$$

$$\omega_t^i = \frac{\lambda_t^i}{\lambda^i} \qquad \text{(Normalized Dynamic Weight)}$$

$$\omega_t = \frac{1}{I}\sum_i \omega_t^i. \qquad \text{(Average Normalized Dynamic Weight)}$$

Aggregate efficiency can be decomposed into (i) exchange efficiency and (ii) production efficiency as

$$\Xi^{AE} = \underbrace{\sum_t \omega_t \sum_j \mathbb{C}ov_i^{\Sigma}\left[MRS_{t,c}^{ij}, \frac{d\chi_{t,c}^{ij}}{d\theta}\right] c_t^j - \sum_t \omega_t \sum_f \mathbb{C}ov_i^{\Sigma}\left[MRS_{t,n}^{if}, \frac{d\chi_{t,n}^{if,s}}{d\theta}\right] n_t^{f,s}}_{\Xi^{AE,X}\ \text{(Exchange Efficiency)}}$$

$$\underbrace{+\sum_t \omega_t \sum_j AMRS_{t,c}^j \frac{dc_t^j}{d\theta} - \sum_t \omega_t \sum_f AMRS_{t,n}^f \frac{dn_t^{f,s}}{d\theta}}_{\Xi^{AE,P}\ \text{(Production Efficiency)}},$$

where the individual and aggregate marginal rates of substitution are given by

$$MRS_{t,c}^{ij} = \frac{\left(\beta^i\right)^t \frac{\partial u_t^i}{\partial c_t^{ij}}}{\lambda_t^i}, \qquad AMRS_{t,c}^j = \sum_i \chi_{t,c}^{ij} MRS_{t,c}^{ij},$$

$$MRS_{t,n}^{if} = -\frac{\left(\beta^i\right)^t \frac{\partial u_t^i}{\partial n_t^{if,s}}}{\lambda_t^i}, \quad AMRS_{t,n}^f = \sum_i \chi_{t,n}^{if,s} MRS_{t,n}^{if},$$

where $\chi_{t,c}^{ij} = c_t^{ij}/c_t^j$ denotes individual i's consumption share of good j at date t, and $\chi_{t,n}^{if,s} =$

$n_t^{if,s}/n_t^{f,s}$ *denotes individual* i*'s factor supply share of factor* f *at date* t*.*

The efficiency component, Ξ^E, corresponds to Kaldor–Hicks efficiency, that is, it is the sum of individual willingness-to-pay for the perturbation in units of the welfare numeraire. The redistribution component, Ξ^{RD}, captures the equity concerns embedded in a particular social welfare function.

The aggregate efficiency component corresponds to the discounted sum — using average dynamic weights as a discount factor — of the aggregate willingness-to-pay for the perturbation. The intertemporal-sharing component captures the contribution to efficiency due to differences in valuation over time across individuals.

The exchange efficiency component corresponds to the efficiency gains associated with the reallocation of consumption and factor supply across individuals. The production efficiency component corresponds to the efficiency gains associated with higher aggregate consumption and lower aggregate factor supply. See Dávila and Schaab (2025) and Dávila and Schaab (2023) for a detailed discussion of the properties of each component.

The production efficiency component, $\Xi^{AE,P}$, is exactly the object characterized in Section B.1, with the aggregate marginal rates of substitution now defined as share-weighted averages of individual marginal rates of substitution. Hence, Theorem 3 applies unchanged to economies with heterogeneous individuals. When $I = 1$, the exchange efficiency, intertemporal-sharing, and redistribution components are zero, and welfare assessments coincide with production efficiency, recovering Lemma 1 in the body of the paper.

B.3 Uncertainty

The results in this appendix extend to economies with uncertainty. Allocations, allocation shares, and welfare-relevant statistics are then indexed by histories, and dynamic weights value consumption across dates and histories. With heterogeneous individuals, the efficiency component of a welfare assessment includes a risk-sharing component, which captures the gains from reallocating consumption risk across individuals, in addition to the aggregate efficiency and intertemporal-sharing components — see Dávila and Schaab (2023). Aggregate efficiency — and with it the production efficiency decomposition — is unchanged in form, now applying date-by-date and history-by-history.

While extending the results is conceptually straightforward, operationalizing the matrix representation under uncertainty involves additional challenges — the propagation matrices acquire a dimension for each history — which we leave for future research.

C Proofs and Derivations

C.1 Sections 3.1 and B.2

Proof of Lemmas 1 and 5. (Welfare Accounting)

Proof. A welfare assessment can be expressed as

$$\frac{dW}{d\theta}=\sum_i \frac{\partial \mathcal{W}}{\partial V^i}\frac{dV^i}{d\theta}=\sum_i \frac{\partial \mathcal{W}}{\partial V^i}\lambda^i\frac{\frac{dV^i}{d\theta}}{\lambda^i}.$$

The normalized welfare assessment takes the form

$$\frac{dW^\lambda}{d\theta}=\frac{\frac{dW}{d\theta}}{\frac{1}{I}\sum_i \frac{\partial \mathcal{W}}{\partial V^i}\lambda^i}=\sum_i \omega^i\frac{\frac{dV^i}{d\theta}}{\lambda^i}=\underbrace{\sum_i \frac{\frac{dV^i}{d\theta}}{\lambda^i}}_{\Xi^E \text{ (Efficiency)}}+\underbrace{\mathbb{C}ov_i^\Sigma\left[\omega^i,\frac{\frac{dV^i}{d\theta}}{\lambda^i}\right]}_{\Xi^{RD} \text{ (Redistribution)}}.$$

Note that the normalized individual welfare gains can be expressed as

$$\frac{dV^{i|\lambda}}{d\theta}=\frac{\frac{dV^i}{d\theta}}{\lambda^i}=\sum_t \underbrace{\frac{\lambda_t^i}{\lambda^i}}_{=\omega_t^i}\frac{dV_t^{i|\lambda}}{d\theta},\quad \text{where}\quad \frac{dV_t^{i|\lambda}}{d\theta}=\sum_j \underbrace{\frac{\left(\beta^i\right)^t\frac{\partial u_t^i}{\partial c_t^{ij}}}{\lambda_t^i}}_{=MRS_{t,c}^{ij}}\frac{dc_t^{ij}}{d\theta}+\sum_f \underbrace{\frac{\left(\beta^i\right)^t\frac{\partial u_t^i}{\partial n_t^{if}}}{\lambda_t^i}}_{=-MRS_{t,n}^{if}}\frac{dn_t^{if,s}}{d\theta}.$$

Hence, the efficiency component can be decomposed as

$$\Xi^E=\sum_t\sum_i \omega_t^i\frac{dV_t^{i|\lambda}}{d\theta}=\underbrace{\sum_t \omega_t\sum_i \frac{dV_t^{i|\lambda}}{d\theta}}_{\Xi^{AE} \text{ (Aggregate Efficiency)}}+\underbrace{\sum_t \omega_t \mathbb{C}ov_i^\Sigma\left[\frac{\omega_t^i}{\omega_t},\frac{dV_t^{i|\lambda}}{d\theta}\right]}_{\Xi^{IS} \text{ (Intertemporal-Sharing)}},$$

where $\omega_t=\frac{1}{I}\sum_i \omega_t^i$ denotes the average dynamic weights. Given that $\chi_{t,c}^{ij}=c_t^{ij}/c_t^j$ and $\chi_{t,n}^{if,s}=n_t^{if,s}/n_t^{f,s}$, we can express changes in individual consumption and factor supply as

$$\frac{dc_t^{ij}}{d\theta}=\frac{d\chi_{t,c}^{ij}}{d\theta}c_t^j+\chi_{t,c}^{ij}\frac{dc_t^j}{d\theta}\quad \text{and}\quad \frac{dn_t^{if,s}}{d\theta}=\frac{d\chi_{t,n}^{if,s}}{d\theta}n_t^{f,s}+\chi_{t,n}^{if,s}\frac{dn_t^{f,s}}{d\theta}.$$

Hence, starting from the definition of aggregate efficiency:

$$
\begin{aligned}
\Xi^{AE} &= \sum_t \omega_t \sum_j \sum_i MRS^{ij}_{t,c} \frac{dc^{ij}_t}{d\theta} - \sum_t \omega_t \sum_f \sum_i MRS^{if}_{t,n} \frac{dn^{if,s}_t}{d\theta} \\
&= \sum_t \omega_t \sum_j \sum_i MRS^{ij}_{t,c} \frac{d\chi^{ij}_{t,c}}{d\theta} c^j_t - \sum_t \omega_t \sum_f \sum_i MRS^{if}_{t,n} \frac{d\chi^{if,s}_{t,n}}{d\theta} n^{f,s}_t \\
&+ \sum_t \omega_t \sum_j \underbrace{\sum_i MRS^{ij}_{t,c} \chi^{ij}_{t,c}}_{=AMRS^j_{t,c}} \frac{dc^j_t}{d\theta} - \sum_t \omega_t \sum_f \underbrace{\sum_i MRS^{if}_{t,n} \chi^{if,s}_{t,n}}_{=AMRS^f_{t,n}} \frac{dn^{f,s}_t}{d\theta} \\
&= \underbrace{\sum_t \omega_t \sum_j \mathbb{C}ov^{\Sigma}_i \left[MRS^{ij}_{t,c}, \frac{d\chi^{ij}_{t,c}}{d\theta} \right] c^j_t - \sum_t \omega_t \sum_f \mathbb{C}ov^{\Sigma}_i \left[MRS^{if}_{t,n}, \frac{d\chi^{if,s}_{t,n}}{d\theta} \right] n^{f,s}_t}_{\Xi^{AE,X} \text{ (Exchange Efficiency)}} \\
&+ \underbrace{\sum_t \omega_t \sum_j AMRS^j_{t,c} \frac{dc^j_t}{d\theta} - \sum_t \omega_t \sum_f AMRS^f_{t,n} \frac{dn^{f,s}_t}{d\theta}}_{\Xi^{AE,P} \text{ (Production Efficiency)}}.
\end{aligned}
$$

Lemma 1 corresponds to the case where $I = 1$ and $F = 0$, which illustrates that $\frac{dW^\lambda}{d\theta} = \Xi^E = \Xi^{AE,P}$ in the body of the paper. □

C.2 Sections 3 and B.1

Proof of Lemma 2. (Capital Inverse Matrix)

Proof. Given (3), we can express $\frac{dk^z_{t+1}}{d\theta}$ as

$$
\frac{dk^z_{t+1}}{d\theta} = \frac{\partial H^z_{t+1}}{\partial \theta} + \sum_j \frac{\partial H^z_{t+1}}{\partial k^{jz,d}_t} \frac{dk^{jz,d}_t}{d\theta} + \sum_j \frac{\partial H^z_{t+1}}{\partial \iota^{zj}_t} \frac{d\iota^{zj}_t}{d\theta}.
$$

Using (9) and (5), we can write

$$
\frac{dk^z_{t+1}}{d\theta} = \frac{\partial H^z_{t+1}}{\partial \theta} + \sum_j \frac{\partial H^z_{t+1}}{\partial k^{jz,d}_t} \left(\frac{d\chi^{jz,d}_{t,k}}{d\theta} k^{z,d}_t + \chi^{jz,d}_{t,k} \frac{dk^z_t}{d\theta} \right) + \sum_j \frac{\partial H^z_{t+1}}{\partial \iota^{zj}_t} \frac{d\iota^{zj}_t}{d\theta}.
$$

Alternatively, in matrix form, we can write

$$
\begin{aligned}
\frac{d\boldsymbol{k}}{d\theta} &= \boldsymbol{H}_\theta + \boldsymbol{H}_k \left(\frac{d\boldsymbol{\chi}_{k^d}}{d\theta} \boldsymbol{k}^d + \boldsymbol{\chi}_{k^d} \frac{d\boldsymbol{k}}{d\theta} \right) + \boldsymbol{H}_\iota \frac{d\boldsymbol{\iota}}{d\theta} \\
&= \boldsymbol{\mathcal{A}}_k \left(\boldsymbol{H}_\theta + \boldsymbol{H}_k \frac{d\boldsymbol{\chi}_{k^d}}{d\theta} \boldsymbol{k}^d + \boldsymbol{H}_\iota \frac{d\boldsymbol{\iota}}{d\theta} \right),
\end{aligned}
\tag{OA4}
$$

where $\boldsymbol{\mathcal{A}}_k = \left(\boldsymbol{I}_{Z(T+1)} - \boldsymbol{H}_k \boldsymbol{\chi}_{k^d} \right)^{-1}$. That corresponds to Equation (11) in the text. □

Proof of Lemmas 3 and 4. (Goods Inverse Matrices)

Proof. Given (OA1) and the goods resource constraint, $y_t^j = y_t^{j,s} + \bar{y}_t^{j,s}(\theta)$, we can write

$$\frac{dy_t^j}{d\theta} = \frac{\partial G_t^j}{\partial \theta} + \sum_\ell \frac{\partial G_t^j}{\partial x_t^{j\ell}} \frac{dx_t^{j\ell}}{d\theta} + \sum_z \frac{\partial G_t^j}{\partial k_t^{jz,d}} \frac{dk_t^{jz,d}}{d\theta} + \sum_f \frac{\partial G_t^j}{\partial n_t^{jf,d}} \frac{dn_t^{jf,d}}{d\theta} + \frac{d\bar{y}_t^{j,s}}{d\theta}.$$

Using (9), (5), and (OA2), we can write

$$\begin{aligned}\frac{dy_t^j}{d\theta} &= \frac{\partial G_t^j}{\partial \theta} + \sum_\ell \frac{\partial G_t^j}{\partial x_t^{j\ell}} \frac{d\xi_t^{j\ell}}{d\theta} y_t^\ell + \sum_\ell \frac{\partial G_t^j}{\partial x_t^{j\ell}} \xi_t^{j\ell} \frac{dy_t^\ell}{d\theta} + \frac{d\bar{y}_t^{j,s}}{d\theta} \\ &+ \sum_z \frac{\partial G_t^j}{\partial k_t^{jz,d}} \frac{d\chi_{t,k}^{jz,d}}{d\theta} k_t^{z,d} + \sum_z \frac{\partial G_t^j}{\partial k_t^{jz,d}} \chi_{t,k}^{jz,d} \frac{dk_t^{z,d}}{d\theta} + \sum_f \frac{\partial G_t^j}{\partial n_t^{jf,d}} \frac{dn_t^{jf,d}}{d\theta}.\end{aligned}$$

Alternatively, in matrix form, using the capital resource constraint, $\frac{d\boldsymbol{k}^d}{d\theta} = \frac{d\boldsymbol{k}^s}{d\theta} + \frac{d\bar{\boldsymbol{k}}^s}{d\theta}$, we can write

$$\frac{d\boldsymbol{y}}{d\theta} = \boldsymbol{G}_\theta + \boldsymbol{G}_x \frac{d\boldsymbol{\xi}}{d\theta}\boldsymbol{y} + \boldsymbol{G}_x \boldsymbol{\xi} \frac{d\boldsymbol{y}}{d\theta} + \boldsymbol{G}_k \frac{d\boldsymbol{\chi}_{k^d}}{d\theta}\boldsymbol{k}^d + \boldsymbol{G}_k \boldsymbol{\chi}_{k^d}\left(\frac{d\boldsymbol{k}^s}{d\theta} + \frac{d\bar{\boldsymbol{k}}^s}{d\theta}\right) + \boldsymbol{G}_n \frac{d\mathring{\boldsymbol{n}}^d}{d\theta} + \frac{d\bar{\boldsymbol{y}}^s}{d\theta}.$$

With capital endowments, Equation (OA4) becomes $\frac{d\boldsymbol{k}^s}{d\theta} = \boldsymbol{\mathcal{A}}_k\left(\boldsymbol{H}_\theta + \boldsymbol{H}_k \frac{d\boldsymbol{\chi}_{k^d}}{d\theta}\boldsymbol{k}^d + \boldsymbol{H}_k \boldsymbol{\chi}_{k^d} \frac{d\bar{\boldsymbol{k}}^s}{d\theta} + \boldsymbol{H}_\iota \frac{d\mathring{\iota}}{d\theta}\right)$, so we can write

$$\begin{aligned}\frac{d\boldsymbol{y}}{d\theta} &= \boldsymbol{G}_\theta + \boldsymbol{G}_x \frac{d\boldsymbol{\xi}}{d\theta}\boldsymbol{y} + \boldsymbol{G}_x \boldsymbol{\xi}\frac{d\boldsymbol{y}}{d\theta} + \boldsymbol{G}_k \frac{d\boldsymbol{\chi}_{k^d}}{d\theta}\boldsymbol{k}^d + \boldsymbol{G}_n \frac{d\mathring{\boldsymbol{n}}^d}{d\theta} + \boldsymbol{G}_k \boldsymbol{\chi}_{k^d}\frac{d\bar{\boldsymbol{k}}^s}{d\theta} + \frac{d\bar{\boldsymbol{y}}^s}{d\theta} \\ &+ \boldsymbol{G}_k \boldsymbol{\chi}_{k^d}\boldsymbol{\mathcal{A}}_k\left(\boldsymbol{H}_\theta + \boldsymbol{H}_k \frac{d\boldsymbol{\chi}_{k^d}}{d\theta}\boldsymbol{k}^d + \boldsymbol{H}_k \boldsymbol{\chi}_{k^d}\frac{d\bar{\boldsymbol{k}}^s}{d\theta} + \boldsymbol{H}_\iota \frac{d\boldsymbol{\chi}_\iota}{d\theta}\boldsymbol{\iota} + \boldsymbol{H}_\iota \boldsymbol{\chi}_\iota \frac{d\boldsymbol{\phi}_\iota}{d\theta}\boldsymbol{y} + \boldsymbol{H}_\iota \boldsymbol{\chi}_\iota \boldsymbol{\phi}_\iota \frac{d\boldsymbol{y}}{d\theta}\right),\end{aligned}$$

which can be written as

$$\begin{aligned}\frac{d\boldsymbol{y}}{d\theta} &= \boldsymbol{\Psi}_y\left(\boldsymbol{G}_\theta + \boldsymbol{G}_x \frac{d\boldsymbol{\xi}}{d\theta}\boldsymbol{y} + \boldsymbol{G}_k \frac{d\boldsymbol{\chi}_{k^d}}{d\theta}\boldsymbol{k}^d + \boldsymbol{G}_n \frac{d\mathring{\boldsymbol{n}}^d}{d\theta} + \boldsymbol{G}_k \boldsymbol{\chi}_{k^d}\frac{d\bar{\boldsymbol{k}}^s}{d\theta} + \frac{d\bar{\boldsymbol{y}}^s}{d\theta}\right) \\ &+ \boldsymbol{\Psi}_k\left(\boldsymbol{H}_\theta + \boldsymbol{H}_k \frac{d\boldsymbol{\chi}_{k^d}}{d\theta}\boldsymbol{k}^d + \boldsymbol{H}_\iota \frac{d\boldsymbol{\chi}_\iota}{d\theta}\boldsymbol{\iota} + \boldsymbol{H}_\iota \boldsymbol{\chi}_\iota \frac{d\boldsymbol{\phi}_\iota}{d\theta}\boldsymbol{y} + \boldsymbol{H}_k \boldsymbol{\chi}_{k^d}\frac{d\bar{\boldsymbol{k}}^s}{d\theta}\right),\end{aligned} \tag{OA5}$$

where $\boldsymbol{\Psi}_y = \left(\boldsymbol{I}_{J(T+1)} - \boldsymbol{G}_k \boldsymbol{\chi}_{k^d}\boldsymbol{\mathcal{A}}_k \boldsymbol{H}_\iota \boldsymbol{\chi}_\iota \boldsymbol{\phi}_\iota - \boldsymbol{G}_x \boldsymbol{\xi}\right)^{-1}$ and $\boldsymbol{\Psi}_k = \boldsymbol{\Psi}_y \boldsymbol{G}_k \boldsymbol{\chi}_{k^d}\boldsymbol{\mathcal{A}}_k$. That corresponds to Equation (OA3). Equation (12) in the main text corresponds to the case without factors other than capital goods, without intermediate inputs, and without endowments. □

Proof of Lemma 6. (Recursive Marginal Social Value)

Proof. Using the fact that $\boldsymbol{\Psi}_y = \boldsymbol{I}_{J(T+1)} + \boldsymbol{\Psi}_y \boldsymbol{G}_k \boldsymbol{\chi}_{k^d}\boldsymbol{\mathcal{A}}_k \boldsymbol{H}_\iota \boldsymbol{\chi}_\iota \boldsymbol{\phi}_\iota + \boldsymbol{\Psi}_y \boldsymbol{G}_x \boldsymbol{\xi} = \boldsymbol{I}_{J(T+1)} +$

$\boldsymbol{\Psi}_k \boldsymbol{H}_\iota \boldsymbol{\chi}_\iota \boldsymbol{\phi}_\iota + \boldsymbol{\Psi}_y \boldsymbol{G}_x \boldsymbol{\xi}$, we can express the marginal social value of goods as

$$\begin{aligned}
\boldsymbol{MSV}_y &= \boldsymbol{\omega AMRS}_c \boldsymbol{\phi}_c \boldsymbol{\Psi}_y \\
&= \boldsymbol{\omega AMRS}_c \boldsymbol{\phi}_c + \boldsymbol{MSV}_k \boldsymbol{H}_\iota \boldsymbol{\chi}_\iota \boldsymbol{\phi}_\iota + \boldsymbol{MSV}_y \boldsymbol{G}_x \boldsymbol{\xi} \\
&= \boldsymbol{\omega AMRS}_c \boldsymbol{\phi}_c + \boldsymbol{MSV}_k \boldsymbol{H}_\iota \boldsymbol{\chi}_\iota \boldsymbol{\phi}_\iota + \boldsymbol{MSV}_y \boldsymbol{G}_x \boldsymbol{\chi}_x \boldsymbol{\phi}_x \\
&= \boldsymbol{\omega AMRS}_c \boldsymbol{\phi}_c + \boldsymbol{AMWP}_\iota \boldsymbol{\phi}_\iota + \boldsymbol{AMWP}_x \boldsymbol{\phi}_x,
\end{aligned}$$

which corresponds to $MSV_{t,y}^j$ in Equation (OA9) in matrix form. Equation (OA9) corresponds to the case without intermediate inputs.

Next, using the fact that $\boldsymbol{\mathcal{A}}_k = \boldsymbol{I}_{Z(T+1)} + \boldsymbol{\mathcal{A}}_k \boldsymbol{H}_k \boldsymbol{\chi}_{k^d}$, we express the marginal social value of capital as

$$\begin{aligned}
\boldsymbol{MSV}_k &= \boldsymbol{\omega AMRS}_c \boldsymbol{\phi}_c \boldsymbol{\Psi}_k \\
&= \boldsymbol{\omega AMRS}_c \boldsymbol{\phi}_c \boldsymbol{\Psi}_y \boldsymbol{G}_k \boldsymbol{\chi}_{k^d} \boldsymbol{\mathcal{A}}_k \\
&= \boldsymbol{MSV}_y \boldsymbol{G}_k \boldsymbol{\chi}_{k^d} \boldsymbol{\mathcal{A}}_k \\
&= \boldsymbol{MSV}_y \boldsymbol{G}_k \boldsymbol{\chi}_{k^d} + \underbrace{\boldsymbol{MSV}_y \boldsymbol{G}_k \boldsymbol{\chi}_{k^d} \boldsymbol{\mathcal{A}}_k}_{=\boldsymbol{MSV}_k} \boldsymbol{H}_k \boldsymbol{\chi}_{k^d} \\
&= \boldsymbol{MSV}_y \boldsymbol{G}_k \boldsymbol{\chi}_{k^d} + \boldsymbol{MSV}_k \boldsymbol{H}_k \boldsymbol{\chi}_{k^d} \\
&= \boldsymbol{AMWP}_k,
\end{aligned}$$

which corresponds to $MSV_{t,k}^z$ in Equation (OA9) in matrix form. □

Proof of Theorems 1 and 3. (Production Efficiency Decomposition)

Proof. We can express production efficiency in matrix form as

$$\Xi^{AE,P} = \boldsymbol{\omega AMRS}_c \frac{d\boldsymbol{c}}{d\theta} - \boldsymbol{\omega AMRS}_n \frac{d\boldsymbol{n}^s}{d\theta}. \tag{OA6}$$

Note that the change in aggregate consumption is given by

$$\frac{d\boldsymbol{c}}{d\theta} = \frac{d\boldsymbol{y}}{d\theta} - \frac{d\boldsymbol{\iota}}{d\theta} - \frac{d\boldsymbol{x}}{d\theta} = \frac{d\boldsymbol{y}}{d\theta} - \left(\boldsymbol{\phi}_\iota \frac{d\boldsymbol{y}}{d\theta} + \boldsymbol{\phi}_x \frac{d\boldsymbol{y}}{d\theta} + \frac{d\boldsymbol{\phi}_\iota}{d\theta}\boldsymbol{y} + \frac{d\boldsymbol{\phi}_x}{d\theta}\boldsymbol{y}\right) = \boldsymbol{\phi}_c \frac{d\boldsymbol{y}}{d\theta} - \frac{d\boldsymbol{\phi}_\iota}{d\theta}\boldsymbol{y} - \frac{d\boldsymbol{\phi}_x}{d\theta}\boldsymbol{y} \quad .$$

Using (OA5), we can write $\frac{d\boldsymbol{c}}{d\theta}$ as

$$\begin{aligned}
\frac{d\boldsymbol{c}}{d\theta} = \boldsymbol{\phi}_c \boldsymbol{\Psi}_y &\left(\boldsymbol{G}_\theta + \boldsymbol{G}_x \frac{d\boldsymbol{\xi}}{d\theta}\boldsymbol{y} + \boldsymbol{G}_k \frac{d\boldsymbol{\chi}_{k^d}}{d\theta}\boldsymbol{k}^d + \boldsymbol{G}_n \frac{d\mathring{\boldsymbol{n}}^d}{d\theta} + \boldsymbol{G}_k \boldsymbol{\chi}_{k^d} \frac{d\bar{\boldsymbol{k}}^s}{d\theta} + \frac{d\bar{\boldsymbol{y}}^s}{d\theta}\right) - \frac{d\boldsymbol{\phi}_\iota}{d\theta}\boldsymbol{y} - \frac{d\boldsymbol{\phi}_x}{d\theta}\boldsymbol{y} \\
&+ \boldsymbol{\phi}_c \boldsymbol{\Psi}_k \left(\boldsymbol{H}_\theta + \boldsymbol{H}_k \frac{d\boldsymbol{\chi}_{k^d}}{d\theta}\boldsymbol{k}^d + \boldsymbol{H}_\iota \frac{d\boldsymbol{\chi}_\iota}{d\theta}\boldsymbol{\iota} + \boldsymbol{H}_\iota \boldsymbol{\chi}_\iota \frac{d\boldsymbol{\phi}_\iota}{d\theta}\boldsymbol{y} + \boldsymbol{H}_k \boldsymbol{\chi}_{k^d} \frac{d\bar{\boldsymbol{k}}^s}{d\theta}\right).
\end{aligned} \tag{OA7}$$

Combining (OA6) and (OA7) yields

$$\Xi^{AE,P} = \overbrace{\underbrace{\boldsymbol{MWP}_x \frac{d\boldsymbol{\chi}_x}{d\theta}\boldsymbol{x}}_{\substack{\text{Cross-Sectional} \\ \text{Intermediate Input Efficiency}}} + \underbrace{(\boldsymbol{AMWP}_x - \boldsymbol{\omega AMRS}_c)\frac{d\boldsymbol{\phi}_x}{d\theta}\boldsymbol{y}}_{\substack{\text{Aggregate} \\ \text{Intermediate Input Efficiency}}}}^{\text{Intermediate Input Efficiency}}$$

$$+ \overbrace{\underbrace{\boldsymbol{MWP}_\iota \frac{d\boldsymbol{\chi}_\iota}{d\theta}\boldsymbol{\iota}}_{\substack{\text{Cross-Sectional} \\ \text{Investment Efficiency}}} + \underbrace{(\boldsymbol{AMWP}_\iota - \boldsymbol{\omega AMRS}_c)\frac{d\boldsymbol{\phi}_\iota}{d\theta}\boldsymbol{y}}_{\substack{\text{Aggregate} \\ \text{Investment Efficiency}}}}^{\text{Investment Efficiency}}$$

$$+ \overbrace{\underbrace{\boldsymbol{MWP}_n \frac{d\boldsymbol{\chi}_{n^d}}{d\theta}\boldsymbol{n}^d}_{\substack{\text{Cross-Sectional} \\ \text{Factor Use Efficiency}}} + \underbrace{(\boldsymbol{AMWP}_n - \boldsymbol{\omega AMRS}_n)\frac{d\boldsymbol{n}^s}{d\theta}}_{\substack{\text{Aggregate} \\ \text{Factor Efficiency}}}}^{\text{Factor Efficiency}}$$

$$+ \overbrace{\underbrace{\boldsymbol{MWP}_k \frac{d\boldsymbol{\chi}_{k^d}}{d\theta}\boldsymbol{k}^d}_{\substack{\text{Cross-Sectional} \\ \text{Capital Use Efficiency}}}}^{\text{Capital Efficiency}} + \underbrace{\boldsymbol{MSV}_k\boldsymbol{H}_\theta}_{\substack{\text{Accumulation} \\ \text{Technology Change}}} + \underbrace{\boldsymbol{MSV}_y\boldsymbol{G}_\theta}_{\substack{\text{Production} \\ \text{Technology Change}}}$$

$$+ \overbrace{\underbrace{\boldsymbol{MSV}_y \frac{d\bar{\boldsymbol{y}}^s}{d\theta}}_{\substack{\text{Good} \\ \text{Endowment Growth}}} + \underbrace{\boldsymbol{AMWP}_n \frac{d\bar{\boldsymbol{n}}^s}{d\theta}}_{\substack{\text{Factor} \\ \text{Endowment Growth}}} + \underbrace{\boldsymbol{AMWP}_k \frac{d\bar{\boldsymbol{k}}^s}{d\theta}}_{\substack{\text{Capital} \\ \text{Endowment Growth}}}}^{\text{Endowment Growth}},$$

where we use $\frac{d\mathring{\boldsymbol{n}}^d}{d\theta} = \frac{d\boldsymbol{\chi}_{n^d}}{d\theta}\boldsymbol{n}^d + \boldsymbol{\chi}_{n^d}\left(\frac{d\boldsymbol{n}^s}{d\theta} + \frac{d\bar{\boldsymbol{n}}^s}{d\theta}\right)$, and the capital endowment term follows from $\boldsymbol{MSV}_y\boldsymbol{G}_k\boldsymbol{\chi}_{k^d} + \boldsymbol{MSV}_k\boldsymbol{H}_k\boldsymbol{\chi}_{k^d} = \boldsymbol{MWP}_k\boldsymbol{\chi}_{k^d} = \boldsymbol{AMWP}_k$. Theorem 1 corresponds to the case without factors other than capital goods, without intermediate inputs, and without endowments. □

Proof of Corollary 1. (Properties of Production Efficiency Decomposition)

Proof. Proceeding item-by-item:

(a) When $J = 1$, $\mathbb{C}ov_j^\Sigma\left[MWP_{t,k}^{jz}, \frac{d\chi_{t,k}^{jz,d}}{d\theta}\right] = 0$, $\forall z$.

(b) When $Z = 1$, $\mathbb{C}ov_z^\Sigma\left[MWP_{t,\iota}^{zj}, \frac{d\chi_{t,\iota}^{zj}}{d\theta}\right] = 0$, $\forall j$.

(c) If all capital goods are specialized at all dates, $\frac{d\chi_{t,k}^{jz,d}}{d\theta} = 0$, $\forall j$, $\forall z$. If all investment uses are specialized at all dates, $\frac{d\chi_{t,\iota}^{zj}}{d\theta} = 0$, $\forall j$, $\forall z$.

(d) When the marginal welfare products of capital are equalized, $\mathbb{C}ov_j^{\Sigma}\left[MWP_{t,k}^{jz}, \frac{d\chi_{t,k}^{jz,d}}{d\theta}\right] = 0,\ \forall z$. When the marginal welfare products of investment are equalized, $\mathbb{C}ov_z^{\Sigma}\left[MWP_{t,\iota}^{zj}, \frac{d\chi_{t,\iota}^{zj}}{d\theta}\right] = 0,\ \forall j$.

(e) When the investment shares are fixed for all goods at all dates, $\frac{d\phi_{t,\iota}^j}{d\theta} = 0,\ \forall j$.

□

C.3 Section E.2

Proof of Theorem 4. (Production Efficiency Conditions)

Proof. Each condition follows from requiring that the corresponding allocative component of Theorem 1 is non-positive for every feasible perturbation of the associated shares. Because the decomposition is additive and each component involves a separate set of shares, the three margins can be considered independently.

First, consider cross-sectional investment efficiency. For goods with $\iota_t^j > 0$, the investment use shares satisfy $\sum_z \chi_{t,\iota}^{zj} = 1$, so any feasible perturbation satisfies $\sum_z \frac{d\chi_{t,\iota}^{zj}}{d\theta} = 0$ and the covariance-sum reduces to

$$\mathbb{C}ov_z^{\Sigma}\left[MWP_{t,\iota}^{zj}, \frac{d\chi_{t,\iota}^{zj}}{d\theta}\right]\iota_t^j = \sum_z MWP_{t,\iota}^{zj}\frac{d\chi_{t,\iota}^{zj}}{d\theta}\iota_t^j.$$

Consider a perturbation that reallocates an investment share $\varepsilon > 0$ from capital good z' towards capital good z'', which is feasible whenever $\chi_{t,\iota}^{z'j} > 0$. Its contribution to production efficiency is $\varepsilon\left(MWP_{t,\iota}^{z''j} - MWP_{t,\iota}^{z'j}\right)\iota_t^j$. No such perturbation is welfare-improving if and only if the marginal welfare products of investment are equalized across all capital goods with positive shares — in which case the common value is $AMWP_{t,\iota}^j = \sum_z \chi_{t,\iota}^{zj} MWP_{t,\iota}^{zj}$ — and weakly lower for capital goods at the corner, which can only receive shares. This is Theorem 4a).

Second, consider aggregate investment efficiency, whose component in Theorem 1 is

$$\left(AMWP_{t,\iota}^j - \omega_t AMRS_{t,c}^j\right)\frac{d\phi_{t,\iota}^j}{d\theta}y_t^j.$$

When $\phi_{t,\iota}^j \in (0,1)$, perturbations of either sign are feasible, so efficiency requires $AMWP_{t,\iota}^j = \omega_t AMRS_{t,c}^j$. When $\phi_{t,\iota}^j = 1$, only $\frac{d\phi_{t,\iota}^j}{d\theta} \leq 0$ is feasible, so efficiency requires $AMWP_{t,\iota}^j \geq \omega_t AMRS_{t,c}^j$. When $\phi_{t,\iota}^j = 0$, only $\frac{d\phi_{t,\iota}^j}{d\theta} \geq 0$ is feasible; since good j is not invested, the investment use shares are unrestricted, so the planner directs the first unit of investment towards its best use and the relevant statistic is $\max_z\left\{MWP_{t,\iota}^{zj}\right\}$, which efficiency requires to be weakly below $\omega_t AMRS_{t,c}^j$. This is Theorem 4b).

Finally, the argument for capital use shares is symmetric to the one for investment use shares. For capital goods with $k_t^{z,d} > 0$, feasible perturbations satisfy $\sum_j \frac{d\chi_{t,k}^{jz,d}}{d\theta} = 0$, and reallocating a share of capital good z from use j' towards use j'' contributes $\varepsilon\left(MWP_{t,k}^{j''z} - MWP_{t,k}^{j'z}\right)k_t^{z,d}$ to production efficiency. Ruling out welfare-improving perturbations requires the marginal welfare products of capital to be equalized across all uses with positive shares — the common value being $AMWP_{t,k}^{z} = \sum_j \chi_{t,k}^{jz,d} MWP_{t,k}^{jz}$ — and weakly lower for uses at the corner. This is Theorem 4c). □

C.4 Section 4

We prove our results in interior economies with intermediate inputs and factors. Formally, firms operating production technology j choose $x_t^{j\ell}$, $k_t^{jz,d}$, and $n_t^{jf,d}$ to maximize their date-t profits, given by

$$\pi_t^j = \max_{\left\{x_t^{j\ell}, k_t^{jz,d}, n_t^{jf,d}\right\}} p_t^j y_t^j - \sum_{\ell} p_t^{\ell} x_t^{j\ell} - \sum_{z} r_t^z k_t^{jz,d} - \sum_{f} w_t^f n_t^{jf,d},$$

where w_t^f denote the price of factor f at date t, and production technologies are given by (OA1).

The optimality conditions can be compactly expressed as

$$\boldsymbol{AMRS}_c = \boldsymbol{p}, \quad \boldsymbol{\omega q H}_{\iota}\boldsymbol{\chi}_{\iota} = \boldsymbol{\omega p}, \quad \boldsymbol{pG}_x\boldsymbol{\chi}_x = \boldsymbol{p}, \quad \boldsymbol{pG}_k\boldsymbol{\chi}_{k^d} = \boldsymbol{r} \quad \text{and} \quad \boldsymbol{pG}_n\boldsymbol{\chi}_{n^d} = \boldsymbol{w}. \tag{OA8}$$

Derivation of Equation (19). (Welfare-Relevant Statistics in Frictionless Competitive Economies)

Proof. The marginal social value of goods is given by

$$\boldsymbol{MSV}_y = \boldsymbol{\omega AMRS}_c\boldsymbol{\phi_c\Psi}_y = \boldsymbol{\omega p\phi}_c\boldsymbol{\Psi}_y.$$

Next, the market value of the change in aggregate consumption, can be expressed as

$$\begin{aligned}
\boldsymbol{p\phi}_c\boldsymbol{\Psi}_y &= \boldsymbol{p\Psi}_y - \boldsymbol{p\phi}_{\iota}\boldsymbol{\Psi}_y - \boldsymbol{p\phi}_x\boldsymbol{\Psi}_y \\
&= \boldsymbol{p} + \underbrace{\boldsymbol{pG}_k\boldsymbol{\chi}_{k^d}}_{=\boldsymbol{r}}\boldsymbol{\mathcal{A}}_k\boldsymbol{H}_{\iota}\boldsymbol{\chi}_{\iota}\boldsymbol{\phi}_{\iota}\boldsymbol{\Psi}_y + \boldsymbol{pG}_x\boldsymbol{\xi\Psi}_y - \boldsymbol{p\phi}_{\iota}\boldsymbol{\Psi}_y - \boldsymbol{p\phi}_x\boldsymbol{\Psi}_y \\
&= \boldsymbol{p} + \boldsymbol{r\mathcal{A}}_k\boldsymbol{H}_{\iota}\boldsymbol{\chi}_{\iota}\boldsymbol{\phi}_{\iota}\boldsymbol{\Psi}_y + \boldsymbol{pG}_x\boldsymbol{\chi}_x\boldsymbol{\phi}_x\boldsymbol{\Psi}_y - \boldsymbol{p\phi}_{\iota}\boldsymbol{\Psi}_y - \boldsymbol{p\phi}_x\boldsymbol{\Psi}_y \\
&= \boldsymbol{p} + \left(\boldsymbol{r\mathcal{A}}_k\boldsymbol{H}_{\iota}\boldsymbol{\chi}_{\iota} - \boldsymbol{p}\right)\boldsymbol{\phi}_{\iota}\boldsymbol{\Psi}_y + \underbrace{\left(\boldsymbol{pG}_x\boldsymbol{\chi}_x - \boldsymbol{p}\right)}_{=0}\boldsymbol{\phi}_x\boldsymbol{\Psi}_y \\
&= \boldsymbol{p} + \left(\boldsymbol{r\mathcal{A}}_k\boldsymbol{H}_{\iota}\boldsymbol{\chi}_{\iota} - \boldsymbol{p}\right)\boldsymbol{\phi}_{\iota}\boldsymbol{\Psi}_y,
\end{aligned}$$

where we used that $\boldsymbol{\Psi}_y = \boldsymbol{I}_{J(T+1)} + \boldsymbol{G}_k\boldsymbol{\chi}_{k^d}\boldsymbol{\mathcal{A}}_k\boldsymbol{H}_{\iota}\boldsymbol{\chi}_{\iota}\boldsymbol{\phi}_{\iota}\boldsymbol{\Psi}_y + \boldsymbol{G}_x\boldsymbol{\xi\Psi}_y$ and the optimality conditions (OA8). Hence, the marginal social value of goods can be expressed as

$$\boldsymbol{MSV}_y = \boldsymbol{\omega p} + \left(\boldsymbol{\omega r\mathcal{A}}_k\boldsymbol{H}_{\iota}\boldsymbol{\chi}_{\iota} - \boldsymbol{\omega p}\right)\boldsymbol{\phi}_{\iota}\boldsymbol{\Psi}_y.$$

Using the Euler equations, we can get the following

$$\omega_{t+1} q_{t+1}^{z} \frac{\partial H_{t+1}^{z}}{\partial k_t^{z,s}} = \sum_{\Delta=1}^{T-t} \omega_{t+\Delta} r_{t+\Delta}^{z} \prod_{m=1}^{\Delta} \frac{\partial H_{t+m}^{z}}{\partial k_{t+m-1}^{z,s}},$$

where $\frac{\partial H_{t+1}^{z}}{\partial k_t^{z,s}} = \sum_j \chi_{t,k}^{jz,d} \frac{\partial H_{t+1}^{z}}{\partial k_t^{jz,s}}$. Next, note that the date-t elements of $\boldsymbol{\omega r \mathcal{A}}_k$ can be expressed as

$$\omega_t r_t^z + \sum_{\Delta=1}^{T-t} \omega_{t+\Delta} r_{t+\Delta}^{z} \prod_{m=1}^{\Delta} \frac{\partial H_{t+m}^{z}}{\partial k_{t+m-1}^{z,s}} = \omega_t r_t^z + \omega_{t+1} q_{t+1}^{z} \frac{\partial H_{t+1}^{z}}{\partial k_t^{z,s}} = \omega_t q_t^z.$$

Consequently, the elements of $\boldsymbol{\omega r \mathcal{A}}_k \boldsymbol{H}_\iota \boldsymbol{\chi}_\iota$ are given by $\omega_{t+1} \sum_z q_{t+1}^{z} \frac{\partial H_{t+1}^{z}}{\partial \iota_t^{zj}} \chi_{t,\iota}^{zj} = \omega_t p_t^j \sum_z \chi_{t,\iota}^{zj} = \omega_t p_t^j$, proving that $\boldsymbol{\omega r \mathcal{A}}_k \boldsymbol{H}_\iota \boldsymbol{\chi}_\iota = \boldsymbol{\omega p}$. Hence, the marginal social value of goods is given by

$$\boldsymbol{MSV}_y = \boldsymbol{\omega p}.$$

Next, the marginal social value of capital goods is given by

$$\boldsymbol{MSV}_k = \boldsymbol{\omega AMRS}_c \boldsymbol{\phi_c \Psi}_k = \boldsymbol{MSV}_y \boldsymbol{G}_k \boldsymbol{\chi}_{k^d} \boldsymbol{\mathcal{A}}_k = \boldsymbol{\omega p G}_k \boldsymbol{\chi}_{k^d} \boldsymbol{\mathcal{A}}_k = \boldsymbol{\omega r \mathcal{A}}_k,$$

which shows that $MSV_{t,k}^z = \omega_t q_t^z$.

The marginal welfare product of capital is given by

$$\begin{aligned} MWP_{t,k}^{jz} &= MSV_{t,y}^{j} \frac{\partial G_t^j}{\partial k_t^{jz,d}} + MSV_{t+1,k}^{z} \frac{\partial H_{t+1}^{z}}{\partial k_t^{jz,d}} \\ &= \omega_t p_t^j \frac{\partial G_t^j}{\partial k_t^{jz,d}} + \omega_{t+1} q_{t+1}^{z} \frac{\partial H_{t+1}^{z}}{\partial k_t^{jz,d}} \\ &= \omega_t r_t^z + \omega_{t+1} q_{t+1}^{z} \frac{\partial H_{t+1}^{z}}{\partial k_t^{jz,d}} \\ \implies MWP_{t,k}^{jz} &= \omega_t q_t^z, \end{aligned}$$

where we used the optimality conditions (OA8). Lastly, the marginal welfare product of investment is given by

$$MWP_{t,\iota}^{zj} = MSV_{t+1,k}^{z} \frac{\partial H_{t+1}^{z}}{\partial \iota_t^{zj}} = \omega_{t+1} q_{t+1}^{z} \frac{\partial H_{t+1}^{z}}{\partial \iota_t^{zj}} = \omega_t p_t^j,$$

where we used the optimality conditions (OA8) in the last step. □

Proof of Theorem 2a). (Production Efficiency in Frictionless Competitive Economies)

Proof. Theorem 2a) follows immediately when Theorem 1 and Equation (19) are combined with Corollary 1. □

Proof of Theorem 2b). (Production and Accumulation Hulten's Theorem)

Proof. Define the following normalized prices

$$\check{p}_t^j = \frac{p_t^j}{\sum_j p_t^j c_t^j}, \quad \check{q}_t^z = \frac{q_t^z}{\sum_j p_t^j c_t^j}, \quad \check{r}_t^z = \frac{r_t^z}{\sum_j p_t^j c_t^j}, \quad \text{and} \quad \check{w}_t^f = \frac{w_t^f}{\sum_j p_t^j c_t^j}.$$

We can then show that the first-order conditions can be written as

$$\boldsymbol{AMRS}_c = \check{\boldsymbol{p}}, \quad \boldsymbol{\omega}\check{\boldsymbol{q}}\boldsymbol{H}_\iota\boldsymbol{\chi}_\iota = \boldsymbol{\omega}\check{\boldsymbol{p}}, \quad \check{\boldsymbol{p}}\boldsymbol{G}_x\boldsymbol{\chi}_x = \check{\boldsymbol{p}}, \quad \check{\boldsymbol{p}}\boldsymbol{G}_k\boldsymbol{\chi}_{k^d} = \check{\boldsymbol{r}} \quad \text{and} \quad \check{\boldsymbol{p}}\boldsymbol{G}_n\boldsymbol{\chi}_{n^d} = \check{\boldsymbol{w}},$$

and that the Euler equations can be written as

$$\begin{aligned} \omega_t\check{q}_t^z &= \omega_t\check{r}_t^z + \omega_{t+1}\check{q}_{t+1}^z\frac{\partial H_{t+1}^z}{\partial k_t^{jz,s}} && \text{for} \quad t<T \\ \omega_T\check{q}_T^z &= \omega_T\check{r}_T^z && \text{for} \quad t=T. \end{aligned}$$

Using the same steps as in the derivation of Equation (19), we can show that

$$MSV_{t,y}^j = \omega_t\check{p}_t^j, \quad MSV_{t,k}^z = \omega_t\check{q}_t^z, \quad \text{and} \quad MWP_{t,\iota}^{zj} = \omega_t\check{p}_t^j.$$

Analogously to Theorem 2, we can express production efficiency as

$$\Xi^{AE,P} = \sum_t \omega_t \sum_z \check{q}_t^z \frac{\partial H_t^z}{\partial\theta} + \sum_t \omega_t \sum_j \check{p}_t^j \frac{\partial G_t^j}{\partial\theta},$$

which can be expressed as

$$\Xi^{AE,P} = \sum_t \omega_t \sum_z \frac{q_t^z k_t^{z,s}}{\sum_j p_t^j c_t^j}\frac{\partial \ln H_t^z}{\partial \ln \vartheta_{t,k}^z}\frac{d\ln\vartheta_{t,k}^z}{d\theta} + \sum_t \omega_t \sum_j \frac{p_t^j y_t^j}{\sum_j p_t^j c_t^j}\frac{\partial \ln G_t^j}{\partial \ln \vartheta_{t,y}^j}\frac{d\ln\vartheta_{t,y}^j}{d\theta}.$$

Using the definitions of the Domar weights, we get Theorem 2b). □

D Applications

D.1 Measuring Accumulation and Production Domar Weights

D.1.1 List of Sectors

We use the following $J = 20$ industries: Agriculture, Forestry, Fishing, and Hunting (NAICS 11); Mining (NAICS 21); Utilities (NAICS 22); Construction (NAICS 23); Manufacturing (NAICS 31-33); Wholesale Trade (NAICS 42); Retail Trade (NAICS 44-45); Transportation and Warehousing (NAICS 48-49); Information (NAICS 51); Finance and Insurance (NAICS 52); Real Estate and Rental and Leasing (NAICS 53); Professional, Scientific, and Technical Services

(NAICS 54); Management of Companies and Enterprises (NAICS 55); Administrative and Waste Management Services (NAICS 56); Educational Services (NAICS 61); Health and Social Assistance (NAICS 62); Arts, Entertainment, and Recreation (NAICS 71); Accommodation and Food Services (NAICS 72); Other Services (except Public Administration) (NAICS 81); Public Administration (NAICS 92).

D.1.2 Data Construction

Data on nominal gross output and aggregate nominal final consumption are taken from the BEA I-O Use tables. For each year we aggregate the BEA industries and commodities to our $J = 20$ two-digit NAICS partition. Nominal gross output of sector j is the "Total Industry Output" row of the Use table summed across the BEA industries that map to j. Aggregate nominal final consumption, $\sum_j p_t^j c_t^j$, is the sum of the BEA "Personal consumption expenditures" column and the three general-government consumption columns of the Use table. Data on nominal capital are taken from the BEA's Fixed Asset tables (3.1E, 3.1S, 3.1I for the 19 private industries; 7.1 for Public Administration). Real aggregate consumption, C_t, is a Törnqvist chain index of real personal consumption expenditures and real government consumption expenditures, anchored at the 2017 nominal aggregate. To construct the Törnqvist chain index, we use NIPA's chain-type quantity indexes (personal consumption expenditures (PCE) quantity index from NIPA Table 2.3.3 and government-consumption quantity index from NIPA Table 3.9.3) and we compute nominal consumption shares using NIPA's annual nominal series (PCE from NIPA Table 2.3.5 and government consumption from NIPA Table 3.9.5).

D.1.3 Deriving the Dynamic Weight

Let Λ_t denote the Lagrange multiplier associated with date-t budget constraint. The first-order condition with respect to c_t^j is then given by

$$\beta^t C_t^{-\gamma} \left(\frac{c_t^j}{C_t} \right)^{-\frac{1}{\sigma}} = \Lambda_t p_t^j.$$

This can be expressed as

$$\beta^t C_t^{1-\gamma} \left(\frac{c_t^j}{C_t} \right)^{\frac{\sigma-1}{\sigma}} = \Lambda_t p_t^j c_t^j,$$

which implies that

$$\Lambda_t \sum_j p_t^j c_t^j = \beta^t C_t^{1-\gamma}.$$

Now recall that

$$\omega_t = \frac{\lambda_t}{\lambda} = \frac{\Lambda_t \sum_j p_t^j c_t^j}{\lambda}.$$

Taking $\lambda = \sum_t \lambda_t$, we get

$$\omega_t = \frac{\beta^t C_t^{1-\gamma}}{\sum_s \beta^s C_s^{1-\gamma}}.$$

D.1.4 Time Variation in Industry Domar Weights

Time variation in the industry-level weights is small for the discounted sums. The dynamic weights spend half of their mass by 1959, roughly three quarters by 1970, and only 5% after 2000, so the discounted sums value the postwar path largely at its early-postwar levels. Formally, replacing each industry's Domar weight path by its unweighted time mean leaves the discounted sum of accumulation Domar weights essentially unchanged (3.701 versus 3.695) and lowers its production counterpart by 4.1% (2.294 versus 2.391). Almost all of the difference reflects Manufacturing, whose total contribution of 1.127 exceeds its flat-path counterpart of 0.942 because the dynamic weights are largest in the decades in which Manufacturing was largest; the mirror image applies to late-rising service industries such as Health (0.099 versus 0.164) and Finance and Insurance (0.117 versus 0.183). Figure 2 should accordingly be read as a description of the ω_t-weighted postwar path, dominated by the early decades, rather than of the present-day economy.

D.1.5 Decomposing Accumulation Domar Weights

Table OA1 reports the weighted accumulation Domar weight $\sum_t \omega_t \mathcal{D}^j_{t,k}$ broken down by industry j (rows) and capital good $z \in \{\text{Equipment}, \text{Structures}, \text{IPP}\}$ (columns). The row totals match column 2 of Table OA2 (within rounding); the column totals show that Structures absorb 79% of the entire accumulation channel, Equipment 17%, and Intellectual Property Products 4%. Figure OA2 plots the same three column totals as time series: $\mathcal{D}^z_{t,k} = \sum_j \mathcal{D}^{jz}_{t,k} = \frac{q^z_t k^z_t}{\sum_j p^j_t c^j_t}$ for $z \in \{\text{Equipment}, \text{Structures}, \text{IPP}\}$, 1947–2024.

In levels, Structures absorb 2.917 of the 3.695 discounted sum of accumulation Domar weights, Equipment 0.632, and Intellectual Property Products 0.146. The accumulation Domar weights of the U.S. economy are dominated by Structures.

This is consistent across industries. Real Estate's accumulation Domar weight of 1.340 is essentially all Structures (1.316, or 98%), which in turn captures the economy's housing stock. Public Administration's 0.981 is also Structures-dominated (0.738, or 75%) — the highways, schools, federal facilities, and military structures — with sizable contributions from government Equipment (0.177, or 18%) and government Intellectual Property Products (0.066, or 7%). The next two largest holders — Transportation and Warehousing and Utilities — are also Structures-dominated: the rail, port, warehouse, power-plant, and grid infrastructure they carry on their books makes Structures three quarters or more of their accumulation weight.

There are some exceptions. Manufacturing is the only large sector with a balanced capital mix — 0.137 Equipment, 0.111 Structures, and 0.037 IPP — reflecting the heavy machinery, plant, and process-control systems that together constitute manufacturing capital. Information is the only sector in which Intellectual Property Products are quantitatively comparable to the other two types (0.030 IPP versus 0.031 Equipment and 0.046 Structures); this is the welfare-weighted footprint of the late-1990s and post-2000 rise of software and digital intangibles, captured at the industry that holds them rather than at the professional-and-technical-services industry that produces much of them. Construction's accumulation Domar weight is the smallest in the table (0.021) and is overwhelmingly Equipment (0.018, or 86%). Mining's 0.097 is 86% Structures because BEA classifies mines, wells, and shafts as structures rather than equipment.

Table OA1: Accumulation Domar Weights by Capital Good

Sector	Equipment	Structures	IPP	Total
Agriculture, Forestry, Fishing, and Hunting	0.040	0.054	0.000	0.094
Mining	0.014	0.083	0.000	0.097
Utilities	0.031	0.129	0.000	0.160
Construction	0.018	0.003	0.000	0.021
Manufacturing	0.137	0.111	0.037	0.285
Wholesale Trade	0.015	0.007	0.000	0.023
Retail Trade	0.013	0.054	0.000	0.067
Transportation and Warehousing	0.073	0.189	0.000	0.262
Information	0.031	0.046	0.030	0.107
Finance and Insurance	0.012	0.018	0.001	0.031
Real Estate and Rental and Leasing	0.023	1.316	0.000	1.340
Professional, Scientific, and Technical Services	0.005	0.004	0.004	0.014
Management of Companies and Enterprises	0.004	0.020	0.000	0.024
Administrative and Waste Management Services	0.003	0.007	0.000	0.011
Educational Services	0.002	0.021	0.001	0.024
Health and Social Assistance	0.011	0.035	0.001	0.046
Arts, Entertainment, and Recreation	0.003	0.014	0.004	0.021
Accommodation and Food Services	0.013	0.028	0.000	0.042
Other Services (except Public Administration)	0.007	0.040	0.001	0.047
Public Administration	0.177	0.738	0.066	0.981
Total	0.632	2.917	0.146	3.695

Note: Accumulation Domar weights $\sum_t \omega_t \mathcal{D}^{jz}_{t,k}$ where $\mathcal{D}^{jz}_{t,k} = \frac{q^z_t k^{jz}_t}{\sum_j p^j_t c^j_t}$. We define $\mathcal{J}$ as the set of 2-digit NAICS industries and $\mathcal{Z} := \{\text{Equipment}, \text{Structures}, \text{Intellectual Property Products}\}$.

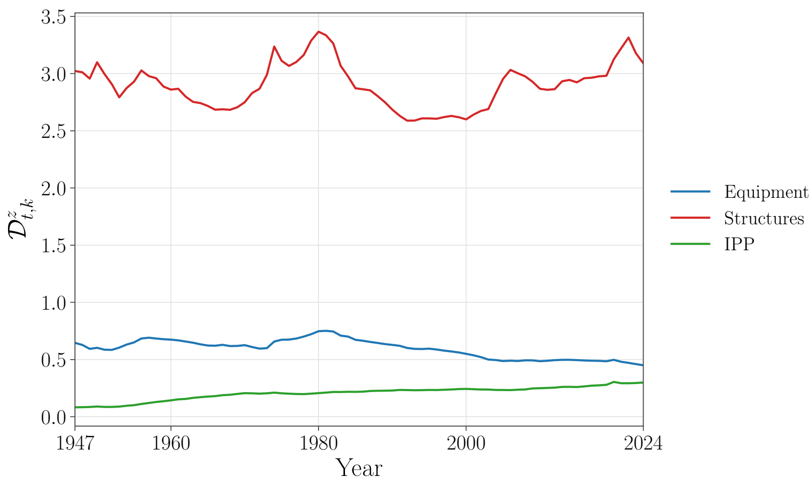


Figure OA2: Application 1: Accumulation Domar Weights by Capital Good $\mathcal{D}^z_{t,k}$

Note: Accumulation Domar weight by capital good $\mathcal{D}^z_{t,k} = \frac{q^z_t k^z_t}{\sum_j p^j_t c^j_t}$ for $z \in \{\text{Equipment}, \text{Structures}, \text{Intellectual Property Products}\}$, 1947–2024.

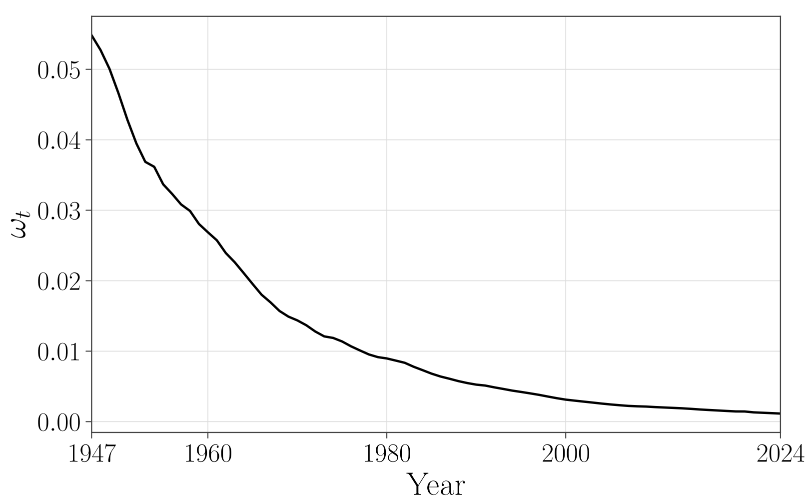


Figure OA3: Application 1: Dynamic Weight

Note: Dynamic weight $\omega_t = \frac{\beta^t C_t^{1-\gamma}}{\sum_s \beta^s C_s^{1-\gamma}}$ under the lifetime welfare numeraire $\lambda = \sum_t \lambda_t$, where C_t denotes real aggregate consumption (Törnqvist chain index), 1947–2024, with $\beta = 0.98$ and $\gamma = 2$.

Table OA2: Production Efficiency Decomposition (Application 1)

Sector	$\sum_t \omega_t \mathcal{D}^j_{t,y}$	$\sum_t \omega_t \mathcal{D}^j_{t,k}$	Total
Agriculture, Forestry, Fishing, and Hunting	0.102	0.094	0.196
Mining	0.051	0.097	0.148
Utilities	0.042	0.160	0.202
Construction	0.134	0.021	0.155
Manufacturing	0.843	0.285	1.127
Wholesale Trade	0.103	0.023	0.125
Retail Trade	0.164	0.067	0.231
Transportation and Warehousing	0.099	0.262	0.362
Information	0.072	0.107	0.179
Finance and Insurance	0.086	0.031	0.117
Real Estate and Rental and Leasing	0.161	1.340	1.501
Professional, Scientific, and Technical Services	0.051	0.014	0.065
Management of Companies and Enterprises	0.028	0.024	0.051
Administrative and Waste Management Services	0.019	0.011	0.030
Educational Services	0.011	0.024	0.035
Health and Social Assistance	0.052	0.046	0.099
Arts, Entertainment, and Recreation	0.013	0.021	0.034
Accommodation and Food Services	0.060	0.042	0.102
Other Services (except Public Administration)	0.053	0.047	0.100
Public Administration	0.247	0.981	1.228
Total	2.391	3.695	6.087

Note: Production efficiency decomposition of permanent Hicks-neutral technology shocks under the lifetime welfare numeraire $\lambda = \sum_t \lambda_t$. We define $\mathcal{J}$ as the set of 2-digit NAICS industries and $\mathcal{Z} := \{\text{Equipment}, \text{Structures}, \text{Intellectual Property Products}\}$ and we compute the accumulation Domar weight of industry j as $\mathcal{D}^j_{t,k} = \frac{\sum_z q^z_t k^{zj}_t}{\sum_j p^j_t c^j_t}$.

D.1.6 Industry Domar Weight Time Series

Figure OA4 reports the accumulation and production Domar weights industry by industry over the postwar period.

The series display substantial reallocation over the postwar period. Two stocks dominate the accumulation weights. Real Estate carries an accumulation Domar weight between 1.3 (1947) and over 1.6 (2024), reflecting the rapid postwar buildup of residential and commercial structures. Public Administration is the second largest accumulation stock throughout the postwar period, with $\mathcal{D}^j_{t,k}$ in the 0.8–1.2 band. Among the remaining capital stocks, Manufacturing, Transportation and Warehousing, Utilities, and Information stand out; each contributes an accumulation Domar weight in the 0.1–0.4 band, with Information rising sharply after the late 1980s. The production Domar weights tell the textbook structural-transformation story: Manufacturing has been on a declining trajectory, while service-producing industries — Finance and Insurance, Real Estate, Professional and Technical Services, and Health — rise steadily over the same period. Agriculture's production weight is small throughout and trends modestly downward.

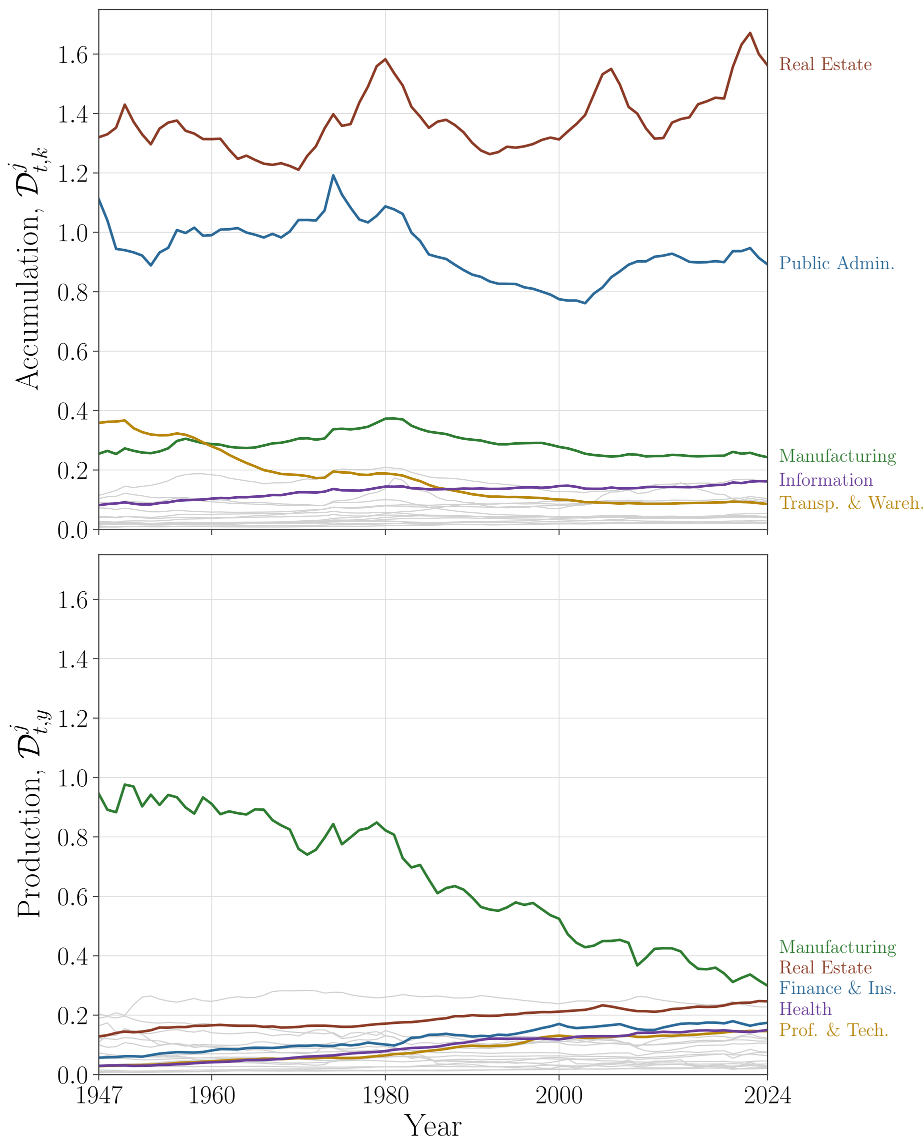


Figure OA4: Application 1: Industry Domar Weights

Note: Accumulation and production Domar weights, $\mathcal{D}^j_{t,k}$ and $\mathcal{D}^j_{t,y}$, by industry (postwar U.S., 1947–2024; $\beta = 0.98$, $\gamma = 2$, $\sigma = 0.349$). Both panels share the same vertical scale. The five series discussed in the text are highlighted and labeled directly at the right edge; the remaining fifteen industries are drawn in gray.

D.2 Origins of Welfare Losses in a Credit Crunch

D.2.1 Equilibrium Characterization

Individual i chooses $\{c_t^i, a_{t+1}^i\}$ to maximize V^i subject to the budget constraint and, as owner of firm i, internalizes that an additional unit of wealth relaxes the collateral constraint by $1/\theta$

units of capital. The first-order conditions are given by

$$\left(\beta^i\right)^t u'\left(c_t^i\right) = \Lambda_t^i \quad \text{and} \quad \Lambda_t^i = \Lambda_{t+1}^i \left(r_{t+1} + 1 - \delta + \frac{\kappa_{t+1}^i}{\theta}\right),$$

where κ_t^i is the shadow value of the collateral constraint per unit of capital and is given by

$$\kappa_t^i = \max\left\{\alpha p_t^i \left(\frac{a_t^i}{\theta}\right)^{\alpha-1} - r_t, 0\right\}.$$

Firm i's capital demand is given by

$$k_t^{i,d} = \min\left\{\frac{a_t^i}{\theta}, \left(\frac{\alpha p_t^i}{r_t}\right)^{\frac{1}{1-\alpha}}\right\}.$$

Final-good profit maximization yields the inverse demands and the ideal price index

$$p_t^i = \zeta_y^i \frac{y_t}{x_t^i} \Longrightarrow p_t^i x_t^i = \zeta_y^i y_t, \quad \text{and} \quad 1 = \prod_{i=1}^{2} \left(\frac{p_t^i}{\zeta_y^i}\right)^{\zeta_y^i}.$$

Because $x_t^i = y_t^i = \left(k_t^{i,d}\right)^\alpha$ in equilibrium, firm i's revenue is $p_t^i y_t^i = \zeta_y^i y_t$ and its marginal revenue product of capital is

$$\alpha p_t^i \left(k_t^{i,d}\right)^{\alpha-1} = \alpha \zeta_y^i \frac{y_t}{k_t^{i,d}}, \quad \text{so that} \quad \kappa_t^i = \max\left\{\alpha \zeta_y^i \frac{y_t}{k_t^{i,d}} - r_t, 0\right\} \quad \text{and} \quad \pi_t^i = \zeta_y^i y_t - r_t k_t^{i,d}.$$

We use date-t aggregate consumption as date-t welfare numeraire and perpetual aggregate consumption as lifetime welfare numeraire, which implies that

$$\lambda_t^i = \Lambda_t^i c_t \quad \text{and} \quad \lambda^i = \sum_t \lambda_t^i.$$

In welfare-numeraire form, the Euler equation reads

$$\omega_t^i = \omega_{t+1}^i \left(\frac{c_t}{c_{t+1}}\right) \left[r_{t+1} + 1 - \delta + \frac{\kappa_{t+1}^i}{\theta}\right].$$

D.2.2 Steady State

Consider a steady state with $\beta^1 > \beta^2$: individual 1 (services) is unconstrained and prices capital, individual 2 (manufacturing) is permanently constrained ($\kappa^2 > 0$). Individual 1's Euler equation pins down the rental rate,

$$1 = \beta^1 \left[1 + r^\star - \delta\right] \Longrightarrow r^\star = \frac{1}{\beta^1} - 1 + \delta.$$

Individual 2's Euler equation, with the constraint binding, pins down the shadow value

$$1 = \beta^2 \left[1 + r^\star - \delta + \frac{\kappa^2}{\theta}\right] \Longrightarrow \kappa^2 = \theta \left(\frac{1}{\beta^2} - \frac{1}{\beta^1}\right) > 0,$$

so each firm equates its marginal revenue product of capital to the effective rental rate $r^1 \equiv r^\star$ and $r^2 \equiv r^\star + \kappa^2 = r^\star + \theta \left(\frac{1}{\beta^2} - \frac{1}{\beta^1}\right)$. Using $\alpha \zeta_y^i \frac{y^\star}{\left(k^{i,d}\right)^\star} = r^i$, the steady-state capital-output ratios are

$$\eta^i \equiv \frac{\left(k^{i,d}\right)^\star}{y^\star} = \frac{\alpha \zeta_y^i}{r^i},$$

and substituting $\left(k^{i,d}\right)^\star = \eta^i y^\star$ into the final-good technology $y^\star = \prod_i \left[\left(k^{i,d}\right)^\star\right]^{\alpha \zeta_y^i}$ gives

$$y^\star = \left[\prod_{i=1}^{2} \left(\eta^i\right)^{\zeta_y^i}\right]^{\frac{\alpha}{1-\alpha}} \quad \text{and} \quad \left(k^{i,d}\right)^\star = \eta^i y^\star.$$

The binding constraint and capital market clearing then deliver the wealth distribution

$$\left(a^2\right)^\star = \theta \eta^2 y^\star \quad \text{and} \quad \left(a^1\right)^\star = \eta^1 y^\star + (1 - \theta)\, \eta^2 y^\star.$$

D.2.3 Calibration

Every parameter is calibrated to postwar U.S. data; unless noted otherwise, a calibrated parameter is the time average of an annual series over 1947–2024.

Real interest rate and the patient discount factor. The date-t real interest rate is measured as the ten-year Treasury constant-maturity yield net of expected inflation (measured as a five-year moving average of PCE inflation) plus a 3% risk premium (which avoids negative values of the real interest rate). The steady-state rental rate is the time average. The patient individual is unconstrained in steady state, so their Euler equation holds with a zero collateral shadow value and pins the discount factor $1 = \beta^1 \left[1 + r^\star - \delta\right] \Longrightarrow \beta^1 = \frac{1}{1+r^\star-\delta} = 0.9937$.

Credit-market tightness. The collateral constraint is the constraint of Moll (2014) with the maximum leverage ratio equal to the reciprocal of the tightness parameter. We adopt his U.S. estimate, itself disciplined by the U.S. external-finance-to-GDP ratio, so that $\theta_0 = \frac{1}{4.15} = 0.2410$.

Impatient discount factor. With the patient services individual pricing capital, the manufacturing individual's collateral constraint binds in steady state, so their external finance is the levered part of their capital ($k^{2,d} - a^2 = (1 - \theta)\, k^{2,d}$), and their Euler equation holds only because the collateral shadow value tops up the return on wealth, making their effective rental rate

$$r^2 = r^\star + \theta \left(\frac{1}{\beta^2} - \frac{1}{\beta^1}\right).$$

Combining these with the steady-state characterization above, the external-finance-to-output is

$$\left(\frac{k^{2,d}-a^{2}}{y}\right)^{\star}=(1-\theta)\,\eta^{2}=\frac{\alpha\zeta_{y}^{2}\,(1-\theta)}{r^{\star}+\theta\left(\frac{1}{\beta^{2}}-\frac{1}{\beta^{1}}\right)}.$$

In these expressions,

$$\eta^{i}=\frac{\alpha\zeta_{y}^{i}}{r^{i}}\quad\text{with}\quad r^{1}=r^{\star}.$$

When $\theta=\theta_0$, we set $\left(\frac{k^{2,d}-a^{2}}{y}\right)^{\star}=2.53$ corresponding to the U.S. value of the measure of Beck, Demirgüç-Kunt, and Levine (2000) — private credit by deposit-money banks and other financial institutions, plus stock-market capitalization, plus private bond-market capitalization, all relative to GDP — which is the same data that disciplines the leverage estimate behind the tightness parameter, making the two credit-market targets internally consistent.

Because the Cobb–Douglas revenue shares are constant, the moment condition solves in closed form,

$$r^{2}=\frac{(1-\theta_{0})\,\alpha\zeta_{y}^{2}}{2.53}=0.0687\Longrightarrow\beta^{2}=\left[\frac{1}{\beta^{1}}+\frac{r^{2}-r^{\star}}{\theta_{0}}\right]^{-1}=0.9390.$$

Initial wealth distribution. The economy starts at the pre-crunch stationary equilibrium: evaluating the steady-state expressions above at the calibrated parameters,

$$a_{0}=a^{\star}\left(\theta_{0}\right)=(19.124,2.332)\,.$$

D.2.4 Numerical Solution Along the Transition

For each θ we solve the perfect-foresight path $\{a_t^i\}_{t=0}^{T}$ as a two-point boundary-value problem with a_0^i given and $a_T^i=(a^i)^{\star}(\theta)$. Given a candidate wealth path with $k_t=\sum_i a_t^i$, the static equilibrium at each date is one of two cases:

1. Both individuals unconstrained: each firm's capital demand satisfies $\alpha\zeta_y^i y_t/k_t^{i,d}=r_t$, so capital is allocated in proportion to the Cobb–Douglas weights and the rental rate follows from capital market clearing,

$$k_{t}^{i,d}=\zeta_{y}^{i}k_{t},\quad y_{t}=k_{t}^{\alpha}\prod_{i=1}^{2}\left(\zeta_{y}^{i}\right)^{\alpha\zeta_{y}^{i}},\quad r_{t}=\alpha\frac{y_{t}}{k_{t}}=\alpha k_{t}^{\alpha-1}\prod_{i=1}^{2}\left(\zeta_{y}^{i}\right)^{\alpha\zeta_{y}^{i}}.$$

2. Individual 2 constrained: $k_t^{2,d}=a_t^2/\theta$, $k_t^{1,d}=k_t-k_t^{2,d}$, and the rental rate is individual 1's marginal revenue product,

$$y_{t}=\prod_{i=1}^{2}\left(k_{t}^{i,d}\right)^{\alpha\zeta_{y}^{i}}\quad\text{and}\quad r_{t}=\alpha\zeta_{y}^{1}\frac{y_{t}}{k_{t}^{1,d}}.$$

The relevant case is selected by checking the shadow values κ_t^i implied by each candidate. The Euler equations are then imposed through the forward update

$$\hat{a}_{t+1}^i = \frac{(1+r_t-\delta)\,\hat{a}_t^i + \pi_t^i - g_t^i\,(\pi_{t+1}^i - a_{t+2}^i)}{1+g_t^i\,(1+r_{t+1}-\delta)} \quad \text{where} \quad g_t^i = \left[\beta^i\left(1+r_{t+1}-\delta+\frac{\kappa_{t+1}^i}{\theta}\right)\right]^{-\frac{1}{\gamma}},$$

with $\hat{a}_0^i = a_0^i$ and the terminal condition above. An equilibrium is a fixed point $\hat{a} = a$, which we compute with a Newton–Krylov solver warm-started from the status quo θ_0 along the θ grid.

E Additional Results

E.1 Recursive Marginal Social Value

Lemma 6. *(Recursive Marginal Social Value) The marginal social values of good j and capital good z available at date t, $MSV_{t,y}^j$ and $MSV_{t,k}^z$, can be expressed as*

$$MSV_{t,y}^j = \phi_{t,c}^j \omega_t AMRS_{t,c}^j + \phi_{t,x}^j AMWP_{t,x}^j + \phi_{t,\iota}^j AMWP_{t,\iota}^j \quad \textit{and} \quad MSV_{t,k}^z = AMWP_{t,k}^z. \tag{OA9}$$

Lemma 6 provides an alternative for Definition 1. It shows that the marginal social values of goods and capital goods equal the share-weighted sum of the social values of their uses. Good j can be either consumed or used as investment or as an intermediate input, and therefore, its social value corresponds to the sum of the value of consuming its consumption share $\phi_{t,c}^j$, the value of using its intermediate share $\phi_{t,x}^j$ in producing other goods, and the value of using its investment share $\phi_{t,\iota}^j$ in accumulating capital goods. Capital good z is not consumed, and therefore, its social value equals the aggregate marginal welfare product of capital. These definitions are recursive because $AMWP_{t,\iota}^j$ and $AMWP_{t,k}^z$ depend on $MSV_{t,y}^j$ and $MSV_{t,k}^z$, respectively.[21]

E.2 Efficiency Conditions

In this section, we exploit Theorem 1 in order to define the conditions that characterize the set of efficient allocations. In deterministic economies with a single individual, welfare assessments are solely driven by production efficiency (Lemma 1), so we adopt the conventional definition of Pareto efficiency: An allocation is efficient when there is no feasible perturbation in which any of the allocative efficiency components is positive. For this exercise, a perturbation is a direct

[21] Note that the marginal social value of capital can be expressed as

$$\boldsymbol{MSV}_k = \boldsymbol{MSV}_y \boldsymbol{G}_k \boldsymbol{\chi}_{k^d} \boldsymbol{\mathcal{A}}_k,$$

where we used that $\boldsymbol{\Psi}_k = \boldsymbol{\Psi}_y \boldsymbol{G}_k \boldsymbol{\chi}_{k^d} \boldsymbol{\mathcal{A}}_k$. Hence, we can express the aggregate marginal welfare product of investment as

$$\boldsymbol{AMWP}_\iota = \boldsymbol{MSV}_y \boldsymbol{G}_k \boldsymbol{\chi}_{k^d} \boldsymbol{\mathcal{A}}_k \boldsymbol{H}_\iota \boldsymbol{\chi}_\iota.$$

change in the feasible allocation selected by a planner, for given technologies. The technical efficiency components of Theorem 1 — accumulation and production technology change — are then identically zero, so production efficiency gains reduce to the three allocative components: cross-sectional investment efficiency, aggregate investment efficiency, and cross-sectional capital use efficiency.

Working with shares makes the set of feasible perturbations transparent. At each date, the planner controls (i) the investment use shares $\chi_{t,\iota}^{zj}$, which satisfy $\sum_z \chi_{t,\iota}^{zj} = 1$ and $\chi_{t,\iota}^{zj} \geq 0$; (ii) the investment shares $\phi_{t,\iota}^{j} \in [0,1]$, whose complement $\phi_{t,c}^{j} = 1 - \phi_{t,\iota}^{j}$ is the consumption share; and (iii) the capital use shares $\chi_{t,k}^{jz,d}$, which satisfy $\sum_j \chi_{t,k}^{jz,d} = 1$ and $\chi_{t,k}^{jz,d} \geq 0$. When a non-negativity constraint binds, only one-sided perturbations are feasible: it is not possible to reallocate investment away from a capital good that receives no investment, or capital away from a technology that uses none. Properly incorporating these constraints is essential for characterizing efficiency conditions in disaggregated economies, where investment networks are sparse and capital goods are specialized. Theorem 4 provides the necessary conditions for production efficiency.[22]

Theorem 4. *(Production Efficiency Conditions) An efficient allocation satisfies, at all dates t:*

(a) *(Cross-Sectional Investment Efficiency) For goods with $\iota_t^j > 0$, it must be that $MWP_{t,\iota}^{zj} = AMWP_{t,\iota}^{j}$, $\forall z$ s.t. $\chi_{t,\iota}^{zj} > 0$; and $MWP_{t,\iota}^{zj} \leq AMWP_{t,\iota}^{j}$, $\forall z$ s.t. $\chi_{t,\iota}^{zj} = 0$.*

(b) *(Aggregate Investment Efficiency) For goods with $y_t^j > 0$, it must be that $\max_z \left\{ MWP_{t,\iota}^{zj} \right\} \leq \omega_t AMRS_{t,c}^{j}$, $\forall j$ s.t. $\phi_{t,\iota}^{j} = 0$; $AMWP_{t,\iota}^{j} = \omega_t AMRS_{t,c}^{j}$, $\forall j$ s.t. $\phi_{t,\iota}^{j} \in (0,1)$; and $AMWP_{t,\iota}^{j} \geq \omega_t AMRS_{t,c}^{j}$, $\forall j$ s.t. $\phi_{t,\iota}^{j} = 1$.*

(c) *(Cross-Sectional Capital Use Efficiency) For capital goods with $k_t^{z,d} > 0$, it must be that $MWP_{t,k}^{jz} = AMWP_{t,k}^{z}$, $\forall j$ s.t. $\chi_{t,k}^{jz,d} > 0$; and $MWP_{t,k}^{jz} \leq AMWP_{t,k}^{z}$, $\forall j$ s.t. $\chi_{t,k}^{jz,d} = 0$.*

Pareto efficiency requires the equalization of $MWP_{t,\iota}^{zj}$ across all capital goods that receive investment of good j, with $MWP_{t,\iota}^{zj}$ potentially lower for capital goods in which good j is not invested. Otherwise, it is feasible and welfare-improving to reallocate investment from low to high marginal welfare product uses, for given aggregate investment ι_t^j. At the corner where capital good z receives no investment of good j, it is not feasible to reallocate investment away from z, even though marginal welfare products are not equalized. The same logic applies to the allocation of capital across uses in Theorem 4c).[23]

Theorem 4b) characterizes the consumption-investment margin. For interior investment shares, the value of investing a marginal unit of good j, $AMWP_{t,\iota}^{j}$, must equal the value of consuming it, $\omega_t AMRS_{t,c}^{j}$ — a generalized Euler equation. Since the marginal welfare products embed the marginal social values of capital, which account for the full propagation of investment

[22]Under standard concavity assumptions on preferences and technologies, the conditions in Theorem 4 are also sufficient.

[23]When $\iota_t^j = 0$, the investment use shares are indeterminate and cross-sectional investment efficiency is vacuous; the relevant condition is the corner condition in Theorem 4b).

through accumulation and production, this condition disciplines the entire path of allocations. At the corner $\phi^j_{t,\iota} = 0$, good j is not invested at all, and efficiency only requires that its best investment use, $\max_z \left\{MWP^{zj}_{t,\iota}\right\}$, is weakly dominated by consumption. Alternatively, at the corner $\phi^j_{t,\iota} = 1$, the individual does not consume good j, and the value of investing must weakly exceed the value of consuming.

Example 6. [Efficiency Conditions in the Neoclassical Growth Model]. In the neoclassical growth model ($J = Z = 1$), conditions a) and c) of Theorem 4 are vacuous, and condition b) recovers the standard Euler equation. With a single capital good, $AMWP_{t,\iota} = MWP_{t,\iota} = MSV_{t+1,k}$, since $\frac{\partial H_{t+1}}{\partial \iota_t} = 1$. At an interior allocation, Lemma 6 yields $MSV_{t+1,y} = \omega_{t+1} AMRS_{t+1,c}$ and $MSV_{t+2,k} = AMWP_{t+1,\iota} = \omega_{t+1} AMRS_{t+1,c}$, so condition b) at dates t and $t+1$ implies

$$\omega_t AMRS_{t,c} = \omega_{t+1} AMRS_{t+1,c} \left(1 - \delta + \frac{\partial G_{t+1}}{\partial k^d_{t+1}}\right) \iff \frac{\partial u_t}{\partial c_t} = \beta \frac{\partial u_{t+1}}{\partial c_{t+1}} \left(1 - \delta + \frac{\partial G_{t+1}}{\partial k^d_{t+1}}\right),$$

where the equivalence uses $\omega_t AMRS_{t,c} = \beta^t \frac{\partial u_t}{\partial c_t} / \lambda$. At the corner $\phi_{t,\iota} = 0$ — investment irreversibility binding — condition b) instead delivers the familiar inequality

$$\frac{\partial u_t}{\partial c_t} \geq \beta \frac{\partial u_{t+1}}{\partial c_{t+1}} \left(1 - \delta + \frac{\partial G_{t+1}}{\partial k^d_{t+1}}\right).$$

Remark 9. *(Interior Economies)* When all shares are interior — $\chi^{zj}_{t,\iota} > 0$, $\phi^j_{t,\iota} \in (0,1)$, and $\chi^{jz,d}_{t,k} > 0$ for all j and z at all dates — Theorem 4 reduces to a system of equalities: marginal welfare products of investment equalized across capital goods, marginal welfare products of capital equalized across uses, and the generalized Euler equation $AMWP^j_{t,\iota} = \omega_t AMRS^j_{t,c}$ for every good. These are the classical efficiency conditions. The inequality conditions become relevant at finer levels of disaggregation, where investment matrices are sparse — each capital good is typically accumulated from few goods — and capital goods are specialized in production. In such economies, marginal welfare products need not be equalized at an efficient allocation, and inferring inefficiency from dispersion in marginal products is unwarranted.

Remark 10. *(Planning Problem and Lagrange Multipliers)* Each allocative efficiency component maps into an optimality condition of the planning problem that maximizes V subject to the production technologies, accumulation technologies, resource constraints, and non-negativity constraints. At an efficient allocation, the marginal social values $MSV^j_{t,y}$ and $MSV^z_{t,k}$ equal the Lagrange multipliers on the date-t resource constraints for good j and capital good z, normalized by λ, and the inequality conditions in Theorem 4 correspond to the complementary slackness conditions on the non-negativity constraints. The decomposition delivers these conditions directly, without setting up the planning problem: away from efficient allocations, the allocative components quantify the welfare gains from violations of each condition separately.

E.3 Welfare Numeraires and Global Assessment

The welfare assessments in the body of the paper are marginal. In this section, we show how to recover Equation (6) from a global consumption-equivalent formulation, how the choice of welfare numeraire determines λ, and how marginal assessments aggregate to global ones. The global assessment of a discrete change from a baseline $\underline{\theta}$ to θ can be expressed as a consumption equivalent (Lucas, 1987), $V^{\lambda}(\theta|\underline{\theta})$, implicitly defined by

$$\sum_t \beta^t u_t \left(\left\{ c_t^j \left(\underline{\theta} \right) \right\}_{j \in \mathcal{J}} \left(1 + V^{\lambda} \left(\theta | \underline{\theta} \right) \right) \right) = \sum_t \beta^t u_t \left(\left\{ c_t^j \left(\theta \right) \right\}_{j \in \mathcal{J}} \right).$$

That is, $V^{\lambda}(\theta|\underline{\theta})$ is the uniform proportional increase in baseline consumption of every good at every date that leaves the individual as well off as the change in θ. Note also that marginal assessments can be integrated to yield global assessments, using $V^{\lambda}(\theta|\underline{\theta}) = \int_{\underline{\theta}}^{\theta} \frac{dV^{\lambda}(\theta'|\underline{\theta})}{d\theta} d\theta'$.

Differentiating both sides with respect to θ and evaluating the derivative at $\theta = \underline{\theta}$, where $V^{\lambda}(\underline{\theta}|\underline{\theta}) = 0$, yields

$$\frac{dV^{\lambda}}{d\theta} = \frac{\frac{dV}{d\theta}}{\lambda}, \quad \text{where} \quad \lambda = \sum_t \beta^t \sum_j \frac{\partial u_t}{\partial c_t^j} c_t^j.$$

That is, the consumption-equivalent formulation selects aggregate perpetual consumption as welfare numeraire. Decomposing $\lambda = \sum_t \lambda_t$, where $\lambda_t = \beta^t \sum_j \frac{\partial u_t}{\partial c_t^j} c_t^j$ is the marginal value of a uniform proportional increase in date-t consumption, recovers Equation (6), with dynamic weights $\omega_t = \frac{\lambda_t}{\lambda}$ that add up to one.

The remaining welfare numeraires considered in the body of the paper follow from modifying the consumption-equivalent formulation. Applying the uniform proportional increase only to date-0 consumption yields $\lambda = \sum_j \frac{\partial u_0}{\partial c_0^j} c_0^j$ (date-0 aggregate consumption numeraire), while an additive increment of $V^{\lambda}(\theta|\underline{\theta})$ units of every good at date 0 yields $\lambda = \sum_j \frac{\partial u_0}{\partial c_0^j}$ (date-0 consumption numeraire).

E.4 Capital Reversibility with Non-negativity Constraints

The neoclassical growth model of Example 1 features investment irreversibility: since investment is a use of the good, $\iota_t \geq 0$, and the stock of capital cannot be run down faster than depreciation. Capital reversibility — the ability to convert installed capital back into consumption — can be represented with two production technologies, using the same device as in Example 5.

The economy features two goods ($J = 2$), perfect substitutes in consumption, with $V = \sum_t \beta^t u_t \left(c_t^1 + c_t^2 \right)$, and a single capital good ($Z = 1$). Technology 1 produces the good, with $y_t^1 = G_t^1 \left(k_t^{1,d} \right)$. Technology 2 disassembles capital, with $y_t^2 = \kappa k_t^{2,d}$, where $\kappa \in (0, 1]$ denotes the scrap recovery rate: $\kappa = 1$ captures full reversibility and $\kappa < 1$ captures scrap losses. Capital allocated to disassembly exits the stock, so the accumulation technology is $k_{t+1} = (1 - \delta) k_t^{1,d} + \iota_t^1$, and resource constraints are $y_t^1 = c_t^1 + \iota_t^1$, $y_t^2 = c_t^2$, and $k_t = k_t^{1,d} + k_t^{2,d}$. Because allocation shares are non-negative, reversibility operates one-sidedly through the disassembly share $\chi_{t,k}^{2,d} \geq 0$.

When disassembly is inactive $\left(\chi_{t,k}^{2,d}=0\right)$, Theorem 4c) only requires $MWP_{t,k}^{2} \leq AMWP_{t,k}$: the scrap value of capital, $\kappa\omega_t AMRS_{t,c}^{2}$, is weakly dominated by its value in production and continuation, so the value of installed capital can exceed what it can be converted back into.

When disassembly is active $\left(\chi_{t,k}^{2,d}>0\right)$, marginal welfare products are equalized across capital uses, which pins the value of installed capital to its scrap value. That is, the non-negativity constraint on the disassembly share — rather than a separate constraint on gross investment — makes capital reversibility one-sided, and the corner conditions of Theorem 4 characterize when running down the capital stock is efficient.

E.5 Time to Build

Time-to-build (Kydland and Prescott, 1982) is nested in the environment of Section 2 by modeling an n-period gestation lag as n capital goods — project stages: investment enters the first stage, projects advance one stage per period, and only the final stage is productive. Formally, with $n=3$, investment in new projects accumulates stage-1 capital, with $k_{t+1}^{(1)}=\iota_t^{(1)}$, and stage advancement operates through linear production technologies that convert each stage into investment for the next one, as anticipated in Remark 1: technology $j=2$ uses stage-1 capital to produce a good that is invested one-for-one into stage-2 capital, so that $k_{t+1}^{(2)}=k_t^{(1),d}$, and analogously $k_{t+1}^{(3)}=(1-\delta)\,k_t^{(3),d}+k_t^{(2),d}$, where only stage-3 capital enters the production technology for the consumption good.

Time-to-build changes welfare accounting in two ways. First, an accumulation impulse to a project m stages from completion reaches the productive stock with an m-period delay, and then decays at the depreciation rate: gestation stages empty out every period, so projects in process are a conveyor, not a stock. Second, the marginal social value of a project rises as it approaches completion, by exactly the foregone discounting and depreciation, so welfare accounting prices goods in process stage by stage.

E.6 Illustrating the Capital Inverse Matrix

In this section, we compute the capital inverse matrix $\boldsymbol{\mathcal{A}}_k=\left(\boldsymbol{I}-\boldsymbol{H}_k\boldsymbol{\chi}_{k^d}\right)^{-1}$, introduced in Lemma 2, in closed form for several of the examples of the paper. Define the $Z\times Z$ *survival matrix* $\boldsymbol{a}_t$, which collects the share-weighted marginal accumulation products of capital uses. Since the accumulation of capital good z depends only on its own uses, $\boldsymbol{a}_t$ is diagonal with entries

$$a_t^z=\sum_j \chi_{t,k}^{jz,d}\frac{\partial H_{t+1}^z}{\partial k_t^{jz,d}}.$$

Since capital supplied at date t only affects capital supplied at date $t+1$, the matrix $\boldsymbol{H}_k\boldsymbol{\chi}_{k^d}$ is a one-period-lag operator, and

$$\boldsymbol{I}-\boldsymbol{H}_k\boldsymbol{\chi}_{k^d} = \begin{pmatrix} \boldsymbol{I} & & & & \\ -\boldsymbol{a}_0 & \boldsymbol{I} & & & \\ & -\boldsymbol{a}_1 & \boldsymbol{I} & & \\ & & \ddots & \ddots & \\ & & & -\boldsymbol{a}_{T-1} & \boldsymbol{I} \end{pmatrix} \implies \boldsymbol{\mathcal{A}}_k = \begin{pmatrix} \boldsymbol{I} & & & & \\ \boldsymbol{a}_0 & \boldsymbol{I} & & & \\ \boldsymbol{a}_1\boldsymbol{a}_0 & \boldsymbol{a}_1 & \boldsymbol{I} & & \\ \vdots & & \ddots & \ddots & \\ \boldsymbol{a}_{T-1}\cdots\boldsymbol{a}_0 & \cdots & & \boldsymbol{a}_{T-1} & \boldsymbol{I} \end{pmatrix},$$

whose (s,t) block for $s \geq t$ is the cumulated survival $\boldsymbol{a}_{s-1}\boldsymbol{a}_{s-2}\cdots\boldsymbol{a}_t$. Three structural properties hold in every example below. First, $\boldsymbol{\mathcal{A}}_k$ is block lower triangular: capital propagates only forward in time. Second, the diagonal blocks are identity matrices: a contemporaneous impulse to the supply of capital passes through one for one. Third, the Neumann series $\boldsymbol{\mathcal{A}}_k = \sum_{n\geq 0}(\boldsymbol{H}_k\boldsymbol{\chi}_{k^d})^n$ terminates after $T+1$ terms: round n collects the knock-on effects that arrive n periods after the impulse.

Neoclassical growth model. In Example 1, with $H_{t+1} = (1-\delta)\,k_t^d + \iota_t$ and a single use ($\chi_{t,k} = 1$), the survival rate is $a_t = 1-\delta$, so

$$\boldsymbol{\mathcal{A}}_k = \begin{pmatrix} 1 & & & & \\ 1-\delta & 1 & & & \\ (1-\delta)^2 & 1-\delta & 1 & & \\ \vdots & & & \ddots & \\ (1-\delta)^T & (1-\delta)^{T-1} & \cdots & 1-\delta & 1 \end{pmatrix},$$

with (s,t) entry $(1-\delta)^{s-t}$: one extra unit of capital at date t raises the supply of capital at date s by its surviving fraction. Two polar cases are instructive. With full depreciation ($\delta = 1$), $\boldsymbol{\mathcal{A}}_k = \boldsymbol{I}$: capital is intertemporally inert and all propagation must come through investment, that is, through the goods inverse matrices. With no depreciation ($\delta = 0$), e.g., land, every entry below the diagonal equals one: impulses to the stock are permanent. With adjustment costs, $H_{t+1} = (1-\delta)\,k_t^d + \Upsilon_t\left(k_t^d, \iota_t\right)$, the survival rate becomes $a_t = 1-\delta+\frac{\partial\Upsilon_t}{\partial k_t^d}$, and the same matrix applies with $1-\delta$ replaced by the cumulated products of a_t.

Investment-specific technological change. In Example 2, the survival matrix is $\boldsymbol{a}_t = \operatorname{diag}\left(1-\delta^1, 1-\delta^2\right)$. For $T = 2$,

$$\boldsymbol{\mathcal{A}}_k = \left(\begin{array}{cc|cc|cc} 1 & 0 & & & & \\ 0 & 1 & & & & \\ \hline 1-\delta^1 & 0 & 1 & 0 & & \\ 0 & 1-\delta^2 & 0 & 1 & & \\ \hline \left(1-\delta^1\right)^2 & 0 & 1-\delta^1 & 0 & 1 & 0 \\ 0 & \left(1-\delta^2\right)^2 & 0 & 1-\delta^2 & 0 & 1 \end{array}\right).$$

The matrix factors capital good by capital good into scalar matrices of the neoclassical form: with own-capital accumulation there is no cross-capital propagation inside $\boldsymbol{\mathcal{A}}_k$; capital goods interact only through goods, that is, through the goods inverse and goods-capital inverse matrices. The investment-specific shock itself enters the accumulation impulse $\boldsymbol{H}_\theta$, not the inverse matrix: $\boldsymbol{\mathcal{A}}_k$ then converts this one-time impulse into the lasting response of the corresponding capital stock.

Invariance to capital reallocation. When both technologies use both capital goods and marginal accumulation products are equalized across uses — e.g., a common depreciation rate, with $k^z_{t+1} = (1-\delta)\sum_j k^{jz,d}_t + \iota^z_t$ — the survival rates satisfy

$$a^z_t = \sum_j \chi^{jz,d}_{t,k}(1-\delta) = 1-\delta,$$

because capital use shares add up to one. The capital inverse matrix is therefore invariant to the allocation of capital across technologies: reallocating capital changes welfare through the goods inverse matrices, never through $\boldsymbol{\mathcal{A}}_k$. The same logic applies to the investment network economy of Example 4, where $\boldsymbol{a}_t = \operatorname{diag}\left(1-\delta^1, \ldots, 1-\delta^J\right)$: the network operates through investment, and thus through the goods inverse matrices, while $\boldsymbol{\mathcal{A}}_k$ remains diagonal. This invariance locates exactly where each economic mechanism lives inside the propagation matrices.

E.7 Examples from Section 2: Continued

Example 1. [Production Efficiency in the Neoclassical Growth Model]. We consider the same environment as in Example 1. By construction, all of the cross-sectional components of allocative efficiency of the production efficiency decomposition are zero, so this economy features only technical efficiency gains and aggregate investment efficiency.

Production efficiency is given by

$$\Xi^{AE,P} = \sum_t \left(AMWP_{t,\iota} - \omega_t AMRS_{t,c}\right)\frac{d\phi_{t,\iota}}{d\theta} y_t + \sum_t MSV_{t,k}\frac{\partial H_t}{\partial \theta} + \sum_t MSV_{t,y}\frac{\partial G_t}{\partial \theta}.$$

The adjustment-cost variant of Example 1 has the same production efficiency decomposition.

Nonetheless, with adjustment costs the accumulation technology component would capture perturbations to the depreciation rate or to the adjustment costs, whereas under linear accumulation it captures only perturbations to the depreciation rate.

Example 2. [Production Efficiency with Investment-Specific Technological Change]. The environment in Example 2 is the simplest to study cross-sectional investment efficiency. By construction, the cross-sectional capital use efficiency component is zero, so this economy features technical efficiency gains as well as investment efficiency gains.

Production efficiency is given by

$$\Xi^{AE,P} = \sum_t \left(AMWP_{t,\iota} - \omega_t AMRS_{t,c}\right) \frac{d\phi_{t,\iota}}{d\theta} y_t + \sum_t \mathbb{C}ov_z^{\Sigma}\left[MWP_{t,\iota}^z, \frac{d\chi_{t,\iota}^z}{d\theta}\right] \iota_t$$
$$+ \sum_t \sum_z MSV_{t,k}^z \frac{\partial H_t^z}{\partial \theta} + \sum_t MSV_{t,y} \frac{\partial G_t}{\partial \theta}.$$

The accumulation technology component captures the welfare implications of investment-specific technology shocks or shocks to the depreciation rates.

Example 3. [Production Efficiency in the Two-Sector Growth Model]. The environment in Example 3 is the simplest to study cross-sectional capital use efficiency. By construction, the cross-sectional investment component is zero, so this economy features technical efficiency gains, aggregate investment efficiency, and cross-sectional capital use efficiency.

Production efficiency is given by

$$\Xi^{AE,P} = \sum_t \left(AMWP_{t,\iota}^1 - \omega_t AMRS_{t,c}^1\right) \frac{d\phi_{t,\iota}^1}{d\theta} y_t^1 + \sum_t \sum_z \mathbb{C}ov_j^{\Sigma}\left[MWP_{t,k}^{jz}, \frac{d\chi_{t,k}^{jz,d}}{d\theta}\right] k_t^{z,d}$$
$$+ \sum_t \sum_z MSV_{t,k}^z \frac{\partial H_t^z}{\partial \theta} + \sum_t \sum_j MSV_{t,y}^j \frac{\partial G_t^j}{\partial \theta}.$$

Example 4. [Production Efficiency in the Investment Network Model]. The environment in Example 4 features specialized capital, since each good is produced using only its own capital good. By construction, the cross-sectional capital use efficiency component is zero, so this economy features technical efficiency gains, aggregate investment efficiency, and cross-sectional investment efficiency.

Production efficiency is given by

$$\Xi^{AE,P} = \sum_t \sum_j \left(AMWP_{t,\iota}^j - \omega_t AMRS_{t,c}^j\right) \frac{d\phi_{t,\iota}^j}{d\theta} y_t^j + \sum_t \sum_j \mathbb{C}ov_z^{\Sigma}\left[MWP_{t,\iota}^{zj}, \frac{d\chi_{t,\iota}^{zj}}{d\theta}\right] \iota_t^j$$
$$+ \sum_t \sum_z MSV_{t,k}^z \frac{\partial H_t^z}{\partial \theta} + \sum_t \sum_j MSV_{t,y}^j \frac{\partial G_t^j}{\partial \theta}.$$

The accumulation technology component captures perturbations to the investment network technologies $\mathfrak{I}_t^j$ or to the depreciation rates $\delta^j(\theta)$.

Example 5. [Production Efficiency in the Credit Cycles Model]. The environment in Example 5 features no investment margin, since land cannot be produced. By construction, both cross-sectional and aggregate investment efficiency are zero, so this economy features technical efficiency gains and cross-sectional capital use efficiency.

Production efficiency is given by

$$\Xi^{AE,P} = \sum_t \mathbb{C}ov_j^{\Sigma}\left[MWP_{t,k}^j, \frac{d\chi_{t,k}^{j,d}}{d\theta}\right] k_t^d + \sum_t \sum_j MSV_{t,y}^j \frac{\partial G_t^j}{\partial \theta},$$

where $\chi_{t,k}^{j,d} = k_t^{j,d}/k_t^d$ denotes the share of land used by technology j and $k_t^d = \bar{k}$. Since the accumulation technology has no primitive parameters, its technology change component is identically zero. For perturbations to policies or constraints — e.g., relaxing the collateral constraint in Kiyotaki and Moore (1997) — production technologies are also unaffected, and all production efficiency gains come from the cross-sectional capital use component, which is positive when land is reallocated toward the use with the higher marginal welfare product.