A Non-Envelope Theorem with Linearly Homogeneous Constraints*

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Abstract

This paper shows that there exists an unambiguous notion of the direct effect of a parameter perturbation on the value of an optimization problem's objective *away from an optimum* for problems with linearly homogeneous constraints. This notion has the interpretation of holding choice variables fixed, and relies on reformulating the optimization problem using *shares* as choice variables. This short paper contains one formal "non-envelope" theorem and four applications to i) consumer demand, ii) cost minimization, iii) planning in exchange economies, and iv) planning in production economies.

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1 Introduction

The envelope theorem is a central tool for comparative statics of unconstrained and constrained optimization problems. It states that, at an optimum, the impact of a parameter perturbation on the value of an optimization problem's objective is solely attributed to its direct effect since optimization ensures that the indirect effect attributed to changes in the choice variables is zero. But what can be said about the impact of a parameter perturbation on the value of the objective *away from an optimum*?

Away from an optimum, the impact of a parameter perturbation on the value of the objective depends on both i) its *direct effect*, defined as the change in the value of the objective *not attributed* to changes in the choice variables, and ii) its *indirect effect*, defined as the change in the value of the objective *attributed* to changes in the choice variables. If a parameter perturbation exclusively impacts the objective function but not the constraints, its direct effect can be unambiguously attributed to the change in the value of the objective that would have ensued if keeping choice variables fixed. However, in general, the split between the direct and indirect effects is ambiguous. In particular, in problems in which the parameter perturbation impacts a binding constraint, it is impossible to define a notion of the direct effect of the perturbation in which the choice variables remain fixed since choice variables must necessarily adjust to satisfy the constraint.[1] In this case, it is possible to define multiple valid notions of the direct (and indirect) effect of the perturbation, as we illustrate in Section 2.3.

The main contribution of this paper is to show that there exists an unambiguous notion of the direct effect (in which choice variables remain fixed) of parameter perturbations for constrained optimization problems with linearly homogeneous constraints, that is, constraints that are homogeneous of degree 1. Since our result characterizes the impact of a parameter perturbation on the value of an optimization problem's objective — although away from an optimum — we refer to it as a "non-envelope theorem", to distinguish it from standard envelope theorems, which only apply at an optimum.

Our result hinges on the fact that Euler's theorem for homogeneous functions makes it possible to reformulate linearly homogeneous constraints as constraints on *shares,* rather than levels. Hence, by performing a change of variables from levels to shares, it is possible to translate a parameter perturbation that impacts a binding constraint into a parameter perturbation that solely impacts the objective function. And in the latter case, the direct effect of the perturbation can be unambiguously attributed to the change in the value of the objective in which choice variables — shares in the reformulated problem — remain fixed.

[1] Formally, if the constraint of an optimization problem is $g(x_1, x_2) = \theta$, for some function $g(\cdot)$ with $\frac{\partial g}{\partial x_1} \neq 0$ and $\frac{\partial g}{\partial x_2} \neq 0$ and a parameter θ, it is impossible to construct perturbations of θ in which the choice variables x_1 and x_2 remain fixed. That is, in response to a $d\theta$ perturbation: $\frac{\partial g}{\partial x_1} dx_1 + \frac{\partial g}{\partial x_2} dx_2 = d\theta$. Hence, whenever $d\theta \neq 0$ it must be that either $dx_1 \neq 0$ or $dx_2 \neq 0$, or both.

The body of this short paper contains one formal theorem and four applications.[2] Theorem 1 formally characterizes the direct and indirect effects of a parameter perturbation on the value of the objective for an optimization problem in which all choice variables are linked via a linearly homogeneous constraint. The usefulness of our contribution lies in whether having a clear notion of the direct effect of a perturbation — that is, the change in the value of the objective induced by a parameter perturbation not attributed to changes in choice variables — has practical value in interesting economic applications. With that goal in mind, we apply the non-envelope theorem result to four economic applications. Our first and second applications consider single-agent optimization problems. In this case, the non-envelope result is useful to study perturbations when agents are behavioral, failing to fully optimize. Our third and fourth applications are planning problems. In this case, the non-envelope result is useful to study perturbations to inefficient allocations.

While the nature of the objective and constraints in each application is different, it is possible to systematically use Theorem 1 to define the direct impact on the value of the objective for any perturbation. The applications highlight that the economic nature of the shares needed to construct the non-envelope result substantially differs across applications (expenditure shares, input shares, consumption shares, and factor use shares), but all follow from the linear homogeneity of the relevant constraints.

Our first application considers a classical demand theory problem, in which a consumer decides how to spend a fixed amount of wealth among many different goods. The non-envelope theorem characterizes the direct change in consumer welfare induced by marginal changes in i) wealth and ii) goods prices, even when consumers are not optimizing. Our second application considers a classical cost minimization problem, in which a firm decides how to choose inputs to minimize the cost of producing a given amount of output. The non-envelope theorem allows us to characterize the direct change in firm costs induced by marginal changes in i) output and ii) factor prices, even when firms are not optimizing. In the consumer case, our result may be particularly helpful in cases in which individual choices do not emerge from maximizing experienced utility, but are determined by rules-of-thumb or other forms of decision utility (Chetty, 2015; Bernheim and Taubinsky, 2018). Similarly, in the cost minimization case, our result can be used to define notions of marginal cost when firms do not make choices that lead to minimizing costs (Heidhues and Kőszegi, 2018).

Our third application considers a planning problem in an exchange economy in which a planner decides how to allocate a fixed amount of many different goods across different individuals. The non-envelope theorem allows us to characterize the direct welfare impact of changes in the aggregate endowment of goods, regardless of whether the allocation of consumption across consumers is

[2] For clarity of exposition, we present the formal result in Section 2 for a problem with a single linearly homogeneous constraint. It is straightforward to extend the logic of our result to problems with multiple linearly homogeneous constraints, as in Applications 3 and 4 in Section 3. See Dávila and Schaab (2023*a*) for how the non-envelope result presented in this paper can be repeatedly applied in a rich model with many linearly homogeneous constraints.

efficient or not. This application illustrates how our result can be used to decompose efficiency gains into gains due to better allocating goods among individuals and gains due to having more of the different goods. Our fourth and final application considers a planning problem in a production economy. In this case, the non-envelope theorem allows us to characterize the direct welfare impact of changes on technology or factor supplies, even when factors of production are misallocated across uses. This type of comparative static exercises is central to the work on factor reallocation and misallocation, as in Hsieh and Klenow (2009), Acemoglu and Restrepo (2018), and Baqaee and Farhi (2020), among many others.

Related literature. Our result is most connected to classical formulations of the envelope theorem, as in Samuelson (1947), Silberberg (1974), or Benveniste and Scheinkman (1979), among others. Envelope theorems are typically stated in mathematical economic textbooks — see, for instance, Simon and Blume (1994), Corbae, Stinchcombe and Zeman (2009), or Sydsaeter et al. (2016) — or in mathematical appendixes, like that of Mas-Colell, Whinston and Green (1995). See Milgrom and Segal (2002) and Sinander (2022) for formulations of the envelope theorem and its converse under minimal assumptions. While it is common to formulate models in terms of shares rather than levels, we are unaware of any existing work showing that it is possible to systematically define an unambiguous notion of the direct effect of a parameter perturbation on the value of an optimization problem by reformulating models with linearly homogeneous constraints in terms of shares.

2 Non-Envelope Theorem

2.1 Optimization Problem

We consider an optimization problem with a finite number $L > 1$ of choice variables, given by $(x_1, \ldots, x_\ell, \ldots, x_L)$. The objective function is denoted by $f : \mathbb{R}^L \to \mathbb{R}$, where V denotes the value of the objective, as in

$$V = f\left(x_1, \ldots, x_\ell, \ldots, x_L; \theta\right). \quad (1)$$

This problem is subject to an equality constraint of the form

$$g\left(x_1, \ldots, x_\ell, \ldots, x_L; \theta\right) = b\left(\theta\right), \quad (2)$$

where $g : \mathbb{R}^L \to \mathbb{R}$, $b : \mathbb{R} \to \mathbb{R}$, and where $\theta \in \mathbb{R}$ denotes a perturbation parameter.[3] We assume that $\frac{\partial g}{\partial \theta} \gtreqless 0$ and $\frac{db}{d\theta} \gtreqless 0$, as well as $\frac{\partial f}{\partial x_\ell} \gtreqless 0$ and $\frac{\partial g}{\partial x_\ell} \neq 0$, $\forall \ell$. We assume that the problem is well-behaved and the solution is interior, although it is straightforward to allow for non-negativity

[3] Our analysis applies unchanged to scenarios with binding inequality constraints. Note that we could have set $b(\theta) = c$, with $c \neq 0$, without loss of generality. Since constraints in many applications take the form $g(\cdot) = b(\theta)$, we have decided to keep the perturbation parameter θ on both sides of the constraint.

constraints. A perturbation of this problem is defined as a change in θ, $d\theta$, and changes in the choice variables, $\frac{dx_1}{d\theta}$ through $\frac{dx_L}{d\theta}$, that satisfy the constraint in (2).

2.2 Envelope Theorem

Standard arguments imply the following optimality conditions:

$$\frac{\frac{\partial f}{\partial x_1}}{\frac{\partial g}{\partial x_1}} = \ldots = \frac{\frac{\partial f}{\partial x_\ell}}{\frac{\partial g}{\partial x_\ell}} = \ldots = \frac{\frac{\partial f}{\partial x_L}}{\frac{\partial g}{\partial x_L}}. \tag{3}$$

Moreover at an optimum, the envelope theorem ensures that the change in the value of the objective induced by a perturbation $d\theta$ is given by

$$\frac{dV}{d\theta} = \frac{\partial f}{\partial \theta} + \frac{\frac{\partial f}{\partial x_\ell}}{\frac{\partial g}{\partial x_\ell}} \left(\frac{db}{d\theta} - \frac{\partial g}{\partial \theta} \right), \ \forall \ell, \tag{4}$$

where optimality in (3) ensures that $\frac{dV}{d\theta}$ can be read off $\frac{\frac{\partial f}{\partial x_\ell}}{\frac{\partial g}{\partial x_\ell}}$ for any choice variable.

The envelope theorem is useful because it establishes that the indirect effect of the perturbation — attributed to the adjustment of choice variables $\frac{dx_1}{d\theta}$ through $\frac{dx_L}{d\theta}$ — is precisely zero. That is, it concludes that none of the change in the value of the objective $\frac{dV}{d\theta}$ is attributed to changes in the choice variables, even when $\frac{dx_1}{d\theta}$ through $\frac{dx_L}{d\theta}$ are non-zero. Therefore, the envelope theorem unambiguously characterizes the direct effect of a perturbation on the value of the objective, given by (4).

2.3 Direct and Indirect Effects Away from an Optimum

But what can be said away from an optimum? Is it possible to unambiguously characterize the contribution to the change in the value of the objective not attributed to changes in the choice variables, that is, the direct effect of a perturbation, perhaps with some qualifications?

In general, the change in the value of the objective induced by a perturbation $d\theta$ for the optimization problem defined in (1)-(2) must satisfy

$$\frac{dV}{d\theta} = \frac{\partial f}{\partial x_1}\frac{dx_1}{d\theta} + \ldots + \frac{\partial f}{\partial x_L}\frac{dx_L}{d\theta} + \frac{\partial f}{\partial \theta}, \tag{5}$$

as well as

$$\frac{\partial g}{\partial x_1}\frac{dx_1}{d\theta} + \ldots + \frac{\partial g}{\partial x_L}\frac{dx_L}{d\theta} + \frac{\partial g}{\partial \theta} = \frac{db}{d\theta}. \tag{6}$$

It should be evident that by solving for any $\frac{dx_\ell}{d\theta}$ in (6) and substituting in (5), it is possible to find L different and equally valid characterizations of $\frac{dV}{d\theta}$. For instance, if we solve for and substitute in $\frac{dx_1}{d\theta}$, we can express $\frac{dV}{d\theta}$ as

$$\frac{dV}{d\theta} = \underbrace{\frac{\partial f}{\partial \theta} + \frac{\frac{\partial f}{\partial x_1}}{\frac{\partial g}{\partial x_1}}\left(\frac{db}{d\theta} - \frac{\partial g}{\partial \theta}\right)}_{\text{direct effect}} + \underbrace{\left(\frac{\frac{\partial f}{\partial x_2}}{\frac{\partial g}{\partial x_2}} - \frac{\frac{\partial f}{\partial x_1}}{\frac{\partial g}{\partial x_1}}\right)\frac{\partial g}{\partial x_2}\frac{dx_2}{d\theta} + \ldots + \left(\frac{\frac{\partial f}{\partial x_L}}{\frac{\partial g}{\partial x_L}} - \frac{\frac{\partial f}{\partial x_1}}{\frac{\partial g}{\partial x_1}}\right)\frac{\partial g}{\partial x_L}\frac{dx_L}{d\theta}}_{\text{indirect effect}}. \tag{7}$$

However, there are many other valid formulations of $\frac{dV}{d\theta}$, each associated with different direct and indirect effects of the perturbation.[4] In fact, any linear combination of the L possible expressions for $\frac{dV}{d\theta}$ after substituting each of x_1 through x_L is equally valid.

At this point, it may seem that it is not possible to make further progress. However, note that whenever $\frac{db}{d\theta} - \frac{\partial g}{\partial \theta} = 0$ (typically, $\frac{db}{d\theta} = \frac{\partial g}{\partial \theta} = 0$), the direct effect of any perturbation is unambiguously given by $\frac{\partial f}{\partial \theta}$. Intuitively, when a perturbation does not directly impact the constraint, it is possible to unambiguously determine what is the change in the value of the objective that would have ensued if choice variables did not change, that is, assuming that $\frac{dx_1}{d\theta} = \ldots = \frac{dx_L}{d\theta} = 0$. Hence, this logic implies that there exists a natural counterpart of the envelope theorem away from an optimum for parameter perturbations that exclusively impact the objective but not the constraints, that is, when $\frac{\partial f}{\partial \theta} \neq 0$ but $\frac{db}{d\theta} - \frac{\partial g}{\partial \theta} = 0$. Therefore, if we can formulate an optimization so that $\frac{db}{d\theta} - \frac{\partial g}{\partial \theta} = 0$, there will be an unambiguous characterization of the direct effect of a parameter perturbation.

2.4 Non-Envelope Theorem

Whenever the constraint function $g(\cdot)$ is linearly homogeneous in the choice variables, i.e., homogeneous of degree 1, it can be reformulated in terms of shares. This change of variables is useful because the constraint of the reformulated problem does not depend on the perturbation parameter θ. This logic allows us to unambiguously define the direct and indirect effects of a perturbation in Theorem 1, which presents the main result of this paper.

Theorem 1. *If the constraint function, $g(\cdot)$, is linearly homogeneous in the choice variables, it is possible to reformulate the optimization problem defined by (1)-(2) in terms of shares ξ_ℓ, given by*

$$\xi_\ell = \frac{\frac{\partial g}{\partial x_\ell}}{b(\theta)} x_\ell, \tag{8}$$

[4]If we instead solve for and substitute in $\frac{dx_L}{d\theta}$, we can express $\frac{dV}{d\theta}$ as

$$\frac{dV}{d\theta} = \underbrace{\frac{\partial f}{\partial \theta} + \frac{\frac{\partial f}{\partial x_L}}{\frac{\partial g}{\partial x_L}}\left(\frac{db}{d\theta} - \frac{\partial g}{\partial \theta}\right)}_{\text{direct}} + \underbrace{\left(\frac{\frac{\partial f}{\partial x_1}}{\frac{\partial g}{\partial x_1}} - \frac{\frac{\partial f}{\partial x_L}}{\frac{\partial g}{\partial x_L}}\right)\frac{\partial g}{\partial x_1}\frac{dx_1}{d\theta} + \ldots + \left(\frac{\frac{\partial f}{\partial x_{L-1}}}{\frac{\partial g}{\partial x_{L-1}}} - \frac{\frac{\partial f}{\partial x_L}}{\frac{\partial g}{\partial x_L}}\right)\frac{\partial g}{\partial x_{L-1}}\frac{dx_{L-1}}{d\theta}}_{\text{indirect}}.$$

ensuring that the constraint of the reformulated problem can be written as

$$\sum_{\ell} \xi_{\ell} = 1. \tag{9}$$

The change in the value of the objective induced by a parameter perturbation can thus be expressed as

$$\frac{dV}{d\theta} = \frac{dV^{non\text{-}\boxtimes}}{d\theta} + \frac{dV^{\xi}}{d\theta},$$

where

$$\frac{dV^{non\text{-}\boxtimes}}{d\theta} = \frac{\partial f}{\partial \theta} + \sum_{\ell} \frac{\frac{\partial f}{\partial x_{\ell}}}{\frac{\partial g}{\partial x_{\ell}}} \sum_{m} \Psi_{\ell m} \left(\xi_m \frac{db(\theta)}{d\theta} - x_m \frac{\partial \left(\frac{\partial g}{\partial x_m} \right)}{\partial \theta} \right) \quad (\textit{direct effect}) \tag{10}$$

$$\frac{dV^{\xi}}{d\theta} = \sum_{\ell} \frac{\frac{\partial f}{\partial x_{\ell}}}{\frac{\partial g}{\partial x_{\ell}}} b(\theta) \sum_{m} \Psi_{\ell m} \frac{d\xi_m}{d\theta}, \quad (\textit{indirect effect}) \tag{11}$$

and where $\Psi_{\ell m}$, defined in (38), is exclusively a function of $\frac{x_{\ell}}{\frac{\partial g}{\partial x_m}} \frac{\partial \left(\frac{\partial g}{\partial x_m} \right)}{\partial x_{\ell}}$, $\forall \ell, m$.

Theorem 1 exploits Euler's homogeneous function theorem to express the constraint in terms of shares, defined in (8). The reformulated constraint in terms of shares does not depend directly on θ. Hence, given the logic outlined above, there exists a unique way of attributing changes in the value of the objective to i) changes in choice variables expressed in shares, defining the indirect effect in (11); and ii) its complement, the direct effect in (10). Intuitively, the direct/non-envelope effect corresponds to the change in the value of the objective induced by a parameter perturbation that is not attributed to changes in the shares ξ_{ℓ}. Through the lens of Theorem 1, the direct/non-envelope effect can be interpreted as the precise combination of the direct effects in the original problem identified above when substituting a single dx_{ℓ} — say in equation (7) — that ensures that shares remain fixed.

When the constraint is not only linearly homogeneous but linear, that is, it can be written as

$$\sum g_{\ell}(\theta) x_{\ell} = b(\theta),$$

where $g_{\ell}(\theta)$ does not depend on any choice variable, $\frac{dV^{\text{non-}\boxtimes}}{d\theta}$ takes the simpler form:

$$\frac{dV^{\text{non-}\boxtimes}}{d\theta} = \frac{\partial f}{\partial \theta} + \sum_{\ell} \frac{\frac{\partial f}{\partial x_{\ell}}}{\frac{\partial g}{\partial x_{\ell}}} \left(\xi_{\ell} \frac{db(\theta)}{d\theta} - x_{\ell} \frac{dg_{\ell}(\theta)}{d\theta} \right). \tag{12}$$

Equation (12) illustrates how the direct effect of tightening or loosening the constraint via $\frac{db(\theta)}{d\theta}$ can be interpreted as a share-weighted average of the direct effects in the original problem when substituting each choice variable at a time, as in (7). Similarly, the direct effect of varying $g_{\ell}(\theta)$

via $\frac{dg_\ell(\theta)}{d\theta}$ can be interpreted as a particular combination of the aforementioned direct effects that ensures that shares remain fixed. The corrections introduced by $\Psi_{\ell m}$ in (11) simply account for the fact that in the general linearly homogeneous case $\frac{\partial g}{\partial x_\ell}$ is in turn a function of x_1 through x_L.

3 Applications

We now apply the non-envelope result to four canonical optimization problems in economics. While the nature of the objectives and constraints in each application is different, it is possible to systematically use Theorem 1 to define the direct impact on the value of the objective for any perturbation. The applications highlight that the economic nature of the *shares* needed to construct the non-envelope result substantially differs across applications (expenditure shares, input shares, consumption shares, and factor use shares), but all follow from the linear homogeneity of the relevant constraints.

Instead of trying to preserve the notation for objective and constraint functions used in Section 2 of this paper, we adopt notation for each application that is closer to how each problem is typically introduced in textbooks, for instance, Mas-Colell, Whinston and Green (1995). We hope that this choice facilitates the exposition.

3.1 Consumer Demand Problem

In our first application, a consumer with initial wealth w chooses a bundle of L goods, c_1 through c_L, to maximize utility

$$V = u\left(c_1, \ldots, c_\ell, \ldots, c_L\right), \tag{13}$$

subject to a budget constraint that is linear in the choice variables

$$p_1 c_1 + \ldots + p_\ell c_\ell + \ldots + p_L c_L = w, \tag{14}$$

where prices p_1 through p_L are taken as given.

Following Theorem 1, it is possible to define expenditure shares for each good as

$$\xi_\ell = \frac{p_\ell c_\ell}{w}.$$

This allows us to reformulate the budget constraint as

$$\sum_\ell \xi_\ell = 1. \tag{15}$$

In this case, the objective function can be written in terms of shares as

$$u\left(\frac{\xi_1 w}{p_1}, \ldots, \frac{\xi_\ell w}{p_\ell} \ldots, \frac{\xi_L w}{p_L}\right). \tag{16}$$

Maximizing (13) subject to (14) is thus equivalent to maximizing (16) subject to (15). In the first formulation, the agent chooses the amount of each good, while in the second one the agent chooses the expenditure shares of each good directly. As explained in Section 2, the key difference between both formulations for our purposes is that in the latter the parameters of the optimization problem are in the objective, rather than the constraint.

Non-Envelope Result. We can directly apply Theorem 1 to express the welfare change of the consumer associated with a general perturbation in which w and p_ℓ change as

$$dV = \underbrace{\sum_\ell \frac{\frac{\partial u}{\partial x_\ell}}{p_\ell}\left(\xi_\ell dw - x_\ell dp_\ell\right)}_{=dV^{\text{non-}\boxtimes}\text{ (direct effect)}} + \underbrace{\sum_\ell \frac{\frac{\partial u}{\partial x_\ell}}{p_\ell} w d\xi_\ell}_{=dV^{\xi}\text{ (indirect effect)}}. \tag{17}$$

The direct/non-envelope term characterizes the contribution of changes in wealth, dw, and prices, dp_ℓ, to the change in the consumer's utility that is not attributed to changes in expenditure shares, $d\xi_\ell$. For changes in wealth, the direct/non-envelope term captures how much the consumer values a unit of wealth when spent according to the expenditure shares ξ_ℓ.[5] Intuitively, consider a scenario in which a consumer receives an extra dollar of wealth. At an optimum, the value of spending the dollar on any good defines the marginal value of the transfer since the consumer is indifferent spending that dollar on any good — that is precisely the condition for optimization, which also yields the envelope theorem. But what if the agent is not optimizing, perhaps because he/she follows a rule-of-thumb or displays other behavioral biases (Chetty, 2015; Bernheim and Taubinsky, 2018)?

Even in those cases, Theorem 1 provides a well-defined notion of the marginal value of the dollar in which the agent spends it respecting expenditure shares. This notion is useful because it ensures that the consumer's behavior, when formulated in terms of expenditure shares, is unchanged, even though the level of consumption and the total expenditure of each good must change with the wealth transfer.

For changes in prices (say of good ℓ), the direct/non-envelope term captures how much the consumer has to change good ℓ's consumption so that the existing expenditure shares remain constant. Once again, the non-envelope result ensures that the consumer's behavior, when formulated in terms of consumption shares, is unchanged in response to changes in prices, even though the level of consumption changes.

[5] While dV in (17) is expressed in utils, it can be trivially translated into money-metric form by dividing by the marginal utility of consuming any good or bundle.

3.2 Cost Minimization Problem

In our second application, a firm that faces (and takes as given) input prices w_1 through w_L chooses a combination of inputs x_1 through x_L to minimize costs

$$\mathcal{C} = w_1 x_1 + \ldots + w_L x_L, \tag{18}$$

subject to a linearly homogeneous production function

$$g\left(x_1, \ldots, x_L\right) = q. \tag{19}$$

Following Theorem 1, it is possible to define input shares for each input

$$\xi_\ell = \frac{\frac{\partial g}{\partial x_\ell} x_\ell}{q}. \tag{20}$$

This allows us to reformulate the production function as

$$\sum_\ell \xi_\ell = 1. \tag{21}$$

With the new formulation in terms of shares, the firm chooses the input share of each input that contributes to total output instead of choosing the quantity of each input.

Non-Envelope Result. We can directly apply Theorem 1 to express the change in total costs for a general perturbation in which w_ℓ and q change as

$$d\mathcal{C} = \underbrace{\sum_\ell x_\ell dw_\ell + \sum_\ell \frac{w_\ell}{\frac{\partial g}{\partial x_\ell}} \sum_m \Psi_{\ell m} \xi_m dq}_{= d\mathcal{C}^{\text{non-}\boxtimes} \text{ (direct effect)}} + \underbrace{q \sum_\ell \frac{w_\ell}{\frac{\partial g}{\partial x_\ell}} \sum_m \Psi_{\ell m} d\xi_m}_{= d\mathcal{C}^{\xi} \text{ (indirect effect)}}, \tag{22}$$

where $\Psi_{\ell m}$ is defined as in (38). Note that the unit of $d\mathcal{C}$ in this expression is the unit in which input prices w_1 through w_L are defined (dollars). The direct/non-envelope term characterizes the contribution of changes in input prices, dw_ℓ, and output, dq, to the change in the firm's total costs that is not attributed to changes in input shares, $d\xi_\ell$. What is the economic interpretation of equation (22)? Consider a scenario in which a firm needs to provide an additional unit of output. At an optimum, the firm is indifferent to increasing output by increasing any input of production. But what if the firm is not optimizing, perhaps because the firm's managers are boundedly rational (Heidhues and Kőszegi, 2018)?

Even when firms do not make choices that lead to minimizing costs, Theorem 1 provides a well-defined notion of the marginal cost of increasing production, or the marginal change in total costs induced by a change in input prices. Similarly to the consumer demand case, these notions are defined to ensure that the contribution of each input to total output, as defined by input shares in (20), remains unchanged.

3.3 Planning Problem in an Exchange Economy

In our third application, we consider a Pareto problem for a planner in an economy with I individuals, indexed by $i = \{1, \ldots, I\}$, who consume L different goods, indexed by $j = \{1, \ldots, L\}$. When $I = L = 2$, this application is an Edgeworth Box economy (Mas-Colell, Whinston and Green, 1995). Formally, we assume that the planner maximizes a utilitarian objective with Pareto weights α_i, so the planner chooses individual i's consumption of good ℓ, $c_{i\ell}$, to maximize the objective

$$W = \sum_i \alpha_i V_i \quad \text{where} \quad V_i = u_i\left(c_{i1}, \ldots, c_{iL}\right), \tag{23}$$

subject to resource constraints for each good ℓ of the form

$$\sum_i c_{i\ell} = y_\ell, \ \forall \ell, \tag{24}$$

where the parameters y_ℓ denote the endowment of each good ℓ.

Following Theorem 1, it is possible to reformulate this problem as maximizing

$$W = \sum_i \alpha_i u_i\left(\xi_{i1} y_1, \ldots, \xi_{iL} y_L\right), \tag{25}$$

where individual i's consumption share of good ℓ is given by

$$\xi_{i\ell} = \frac{c_{i\ell}}{y_\ell}.$$

This in turn allows us to reformulate the resource constraint for each good ℓ as

$$\sum_i \xi_{i\ell} = 1, \ \forall \ell. \tag{26}$$

Minimizing (23) subject to (24) is thus equivalent to minimizing (25) subject to (26). In the first formulation, the planner chooses the quantity of each good allocated to each individual, while in the second one the planner instead chooses the share of aggregate consumption of each good allocated to each individual.

Non-Envelope Result We can directly apply Theorem 1 to express the welfare change induced by a general perturbation in which any y_ℓ changes as

$$dW = \underbrace{\sum_i \alpha_i \sum_\ell \frac{\partial u_i}{\partial c_{i\ell}} \xi_{i\ell} dy_\ell}_{=dW^{\text{non-}\boxtimes} \text{ (direct effect)}} + \underbrace{\sum_i \alpha_i \sum_\ell \frac{\partial u_i}{\partial c_{i\ell}} d\xi_{i\ell} y_\ell}_{=dW^{\xi} \text{ (indirect effect)}}. \tag{27}$$

In this case, the direct/non-envelope term characterizes the contribution of endowment changes to the change in social welfare that is not attributed to changes in consumption shares. Since a social welfare function, such as the one in (23) conflates both efficiency and redistribution considerations, it is useful to separate both, as explained in Dávila and Schaab (2023*b*). If we choose a common unit to aggregate individual welfare gains and losses (welfare numeraire or money-metric), it is straightforward to write the efficiency gains (defined as aggregate willingness to pay in the common unit) implied by (27) as

$$E \equiv \text{Efficiency Gains} = \underbrace{\sum_\ell \left(\sum_i \xi_{i\ell} MRS_{i\ell}\right) dy_\ell}_{=E^{\text{non-}\boxtimes} \text{ (direct effect)}} + \underbrace{\sum_\ell \mathbb{C}ov_i^{\Sigma}\left[MRS_{i\ell}, d\xi_{i\ell}\right] y_\ell}_{=E^{\xi} \text{ (indirect effect)}},$$

where $MRS_{i\ell} = \frac{\frac{\partial u_i}{\partial c_{i\ell}}}{\lambda_i}$ denotes the marginal valuation that individual i attaches to a unit of good ℓ, expressed in the common unit, and where $\mathbb{C}ov_i^{\Sigma}[\cdot,\cdot] = I \cdot \mathbb{C}ov_i[\cdot,\cdot]$ is a cross-sectional covariance-sum across individuals.[6]

Hence, Theorem 1 provides a well-defined notion of the efficiency gains induced by a marginal change in endowments — even at inefficient allocations, in which marginal rates of substitution between goods are not equalized across individuals. It shows that such gains are due to i) a direct effect, which captures the aggregate gain from allocating endowment changes to different individuals in proportion to their consumption shares $\xi_{i\ell}$, and ii) an indirect effect, which captures the reallocation of consumption shares to individuals with different valuations for the goods ($MRS_{i\ell}$).

3.4 Planning Problem in a Production Economy

In our final application, we consider a planning problem in an economy with a single individual who consumes L different goods, indexed by $j = \{1, \ldots, L\}$, which are in turn produced using F factors, indexed by $f = \{1, \ldots, F\}$. Formally, we assume that the planner chooses the allocation of factors to maximize the utility of the single individual

$$W = u\left(c_1, \ldots, c_\ell, \ldots, c_L\right), \tag{28}$$

[6]The denominator λ_i is an individual normalizing factor to express welfare gains in a given common unit.

where production of each good is potentially a function of the F factors, as in

$$c_\ell = z_\ell f_\ell \left(n_{\ell 1}, \ldots, n_{\ell f}, \ldots, n_{\ell F}\right), \; \forall \ell, \tag{29}$$

and where the (predetermined) supply of each factor, n_f, must be allocated across the different uses according to the resource constraints

$$\sum_\ell n_{\ell f} = n_f, \; \forall f. \tag{30}$$

Following Theorem 1, it is possible to reformulate this problem as maximizing (28), where

$$c_\ell = z_\ell f_\ell \left(\xi_{\ell 1} n_1, \ldots, \xi_{\ell f} n_f, \ldots, \xi_{\ell F} n_F\right), \; \forall \ell, \tag{31}$$

where the share of factor f used to produce good ℓ is given by $\xi_{\ell f} = \frac{n_{\ell f}}{n_f}$, which allows us to reformulate the resource constraints for each factor f as

$$\sum_\ell \xi_{\ell f} = 1, \; \forall f. \tag{32}$$

Maximizing (28) subject to (29) and (30) is equivalent to doing so subject to (31) and (32). In the first formulation, the planner chooses the quantity of each factor allocated to producing each good, while in the second one the planner instead chooses the share of each factor allocated to its different uses.

Non-Envelope Result We can directly apply Theorem 1 to express the welfare change induced by a general perturbation in which any z_ℓ or n_ℓ change as

$$dW = \underbrace{\sum_\ell \frac{\partial u}{\partial c_\ell} \left(dz_\ell f_\ell + z_\ell \sum_f \frac{\partial f_\ell}{\partial n_{\ell f}} \xi_{\ell f} dn_f \right)}_{= dW^{\text{non-}\boxtimes} \text{ (direct effect)}} + \underbrace{\sum_\ell \frac{\partial u}{\partial c_\ell} z_\ell \sum_f \frac{\partial f_\ell}{\partial n_{\ell f}} d\xi_{\ell f} n_f}_{= dW^{\xi} \text{ (indirect effect)}}. \tag{33}$$

In this case, the direct/non-envelope term characterizes the contribution of changes in technology or factor endowments not attributed to changes in factor use shares. For changes in technology, the direct/non-envelope term captures the welfare gain induced by having more output, holding factor use shares $\xi_{\ell f}$ constant. For changes in factor endowments, the direct/non-envelope term captures the welfare gain induced by using factors in proportion to their existing uses, again holding factor use shares $\xi_{\ell f}$ constant. Hence, Theorem 1 provides a well-defined notion of the direct welfare effect of a change in technology or factor endowments — even at inefficient allocations, in which factors of production are misallocated across uses.

Note that the indirect effect in this case can be written as

$$\sum_f \mathbb{C}ov_\ell^\Sigma \left[\frac{\partial u}{\partial c_\ell} z_\ell \frac{\partial f_\ell}{\partial n_{\ell f}}, d\xi_{\ell f} \right] n_f, \tag{34}$$

where $\mathbb{C}ov_\ell^\Sigma [\cdot,\cdot] = L \cdot \mathbb{C}ov_\ell [\cdot,\cdot]$ is a cross-sectional covariance-sum across produced goods. Equation (34) shows that the indirect effect captures the reallocation of factors towards uses that increase marginal utility adjusted marginal products ($\frac{\partial u}{\partial c_\ell} z_\ell \frac{\partial f_\ell}{\partial n_{\ell f}}$). This second term is only non-zero for inefficient allocations. The type of comparative static exercises in equation (34) are central to the work on factor reallocation and misallocation, as in Hsieh and Klenow (2009), Acemoglu and Restrepo (2018), and Baqaee and Farhi (2020), among many others.

4 Conclusion

This paper has shown that there exists an unambiguous notion of the direct effect of a parameter perturbation on the value of an optimization problem's objective *away from an optimum* for problems with linearly homogeneous constraints. This "non-envelope" notion has the interpretation of holding choice variables fixed, and relies on reformulating the optimization problem using *shares* as choice variables.

As shown through four canonical applications exploring single-agent and planning problems, it is possible to derive clear insights by systematically applying the non-envelope notion to any optimization problem with linearly homogeneous constraints. We hope that new, meaningful economic applications get developed around the non-envelope result presented in this paper.

Appendix

A Proof of Theorem 1

Proof. The proof of this result is constructive. If the function $g(\cdot)$ is homogeneous of degree 1 in the choice variables, equation (2) can be expressed, appealing to Euler's homogeneous function theorem, as

$$\sum_{\ell} \frac{\partial g}{\partial x_\ell} x_\ell = b(\theta),$$

which allows us to rewrite the constraint in (2) as in (9), where the shares ξ_ℓ are defined in (8). The change in the value of the objective can be expressed as

$$\frac{dV}{d\theta} = \sum_{\ell} \frac{\partial f}{\partial x_\ell} \frac{dx_\ell}{d\theta} + \frac{\partial f}{\partial \theta}.$$

Since (8) implies that $x_\ell = \frac{b(\theta)}{\frac{\partial g}{\partial x_\ell}} \xi_\ell$, it follows that

$$\frac{dx_\ell}{d\theta} = \frac{1}{\frac{\partial g}{\partial x_\ell}} \left(\xi_\ell \frac{db(\theta)}{d\theta} - x_\ell \frac{d\left(\frac{\partial g}{\partial x_\ell}\right)}{d\theta} + b(\theta) \frac{d\xi_\ell}{d\theta} \right), \tag{35}$$

where

$$\frac{d\left(\frac{\partial g}{\partial x_\ell}\right)}{d\theta} = \frac{\partial\left(\frac{\partial g}{\partial x_\ell}\right)}{\partial\theta} + \sum_m \frac{\partial\left(\frac{\partial g}{\partial x_\ell}\right)}{\partial x_m} \frac{dx_m}{d\theta} = \frac{\partial^2 g}{\partial x_\ell \partial\theta} + \sum_m \frac{\partial^2 g}{\partial x_\ell x_m} \frac{dx_m}{d\theta}. \tag{36}$$

Note that when $x_\ell = \xi_\ell = 0$, it must be that $\frac{dx_\ell}{d\theta} = \frac{b(\theta)}{\frac{\partial g}{\partial x_\ell}} \frac{d\xi_\ell}{d\theta}$. Hence, combining (35) and (36), we find that

$$\begin{aligned} \frac{dx_\ell}{d\theta} \frac{\partial g}{\partial x_\ell} &= \xi_\ell \frac{db(\theta)}{d\theta} - x_\ell \frac{\partial\left(\frac{\partial g}{\partial x_\ell}\right)}{\partial\theta} - x_\ell \sum_m \frac{\partial\left(\frac{\partial g}{\partial x_\ell}\right)}{\partial x_m} \frac{dx_m}{d\theta} + b(\theta) \frac{d\xi_\ell}{d\theta} \\ &= \xi_\ell \frac{db(\theta)}{d\theta} - x_\ell \frac{\partial\left(\frac{\partial g}{\partial x_\ell}\right)}{\partial\theta} - \sum_m \frac{x_\ell}{\frac{\partial g}{\partial x_m}} \frac{\partial\left(\frac{\partial g}{\partial x_m}\right)}{\partial x_\ell} \frac{dx_m}{d\theta} \frac{\partial g}{\partial x_m} + b(\theta) \frac{d\xi_\ell}{d\theta}, \end{aligned} \tag{37}$$

where we use the symmetry of second derivatives, that is, $\frac{\partial\left(\frac{\partial g}{\partial x_\ell}\right)}{\partial x_m} = \frac{\partial\left(\frac{\partial g}{\partial x_m}\right)}{\partial x_\ell}$. Hence, we can write (37) as

$$\boldsymbol{X} = \boldsymbol{A} - \boldsymbol{B}\boldsymbol{X},$$

where $\boldsymbol{X}$ and $\boldsymbol{A}$ are $L \times 1$ vectors, and $\boldsymbol{B}$ is a $L \times L$ matrix given by

$$\boldsymbol{X} = \begin{pmatrix} \vdots \\ \frac{dx_\ell}{d\theta}\frac{\partial g}{\partial x_\ell} \\ \vdots \end{pmatrix}, \quad \boldsymbol{A} = \begin{pmatrix} \vdots \\ \xi_\ell \frac{db(\theta)}{d\theta} - x_\ell \frac{\partial\left(\frac{\partial g}{\partial x_\ell}\right)}{\partial \theta} + b\left(\theta\right)\frac{d\xi_\ell}{d\theta} \\ \vdots \end{pmatrix}, \quad \boldsymbol{B} = \begin{pmatrix} \ddots & \vdots & \iddots \\ & \frac{x_\ell}{\frac{\partial g}{\partial x_m}} \frac{\partial\left(\frac{\partial g}{\partial x_m}\right)}{\partial x_\ell} & \\ \iddots & \vdots & \ddots \end{pmatrix}.$$

Hence, $\boldsymbol{X}$ is given by

$$\boldsymbol{X} = \boldsymbol{\Psi}\boldsymbol{A}, \quad \text{where} \quad \boldsymbol{\Psi} = \left(\boldsymbol{I} + \boldsymbol{B}\right)^{-1}, \tag{38}$$

which implies that we can express $\frac{dx_\ell}{d\theta}\frac{\partial g}{\partial x_\ell}$ as

$$\frac{dx_\ell}{d\theta}\frac{\partial g}{\partial x_\ell} = \sum_m \Psi_{\ell m} A_m,$$

where $\Psi_{\ell m}$ is the (ℓ, m) element of $\boldsymbol{\Psi}$ and A_m is the m'th element of $\boldsymbol{A}$. Therefore, we can express $\frac{dV}{d\theta}$ as

$$\frac{dV}{d\theta} = \frac{\partial f}{\partial \theta} + \sum_\ell \frac{\frac{\partial f}{\partial x_\ell}}{\frac{\partial g}{\partial x_\ell}} \sum_m \Psi_{\ell m} \left(\xi_m \frac{db\left(\theta\right)}{d\theta} - x_m \frac{\partial\left(\frac{\partial g}{\partial x_m}\right)}{\partial \theta} + b\left(\theta\right)\frac{d\xi_m}{d\theta}\right),$$

or separating direct and indirect effects as

$$\frac{dV}{d\theta} = \underbrace{\frac{\partial f}{\partial \theta} + \sum_\ell \frac{\frac{\partial f}{\partial x_\ell}}{\frac{\partial g}{\partial x_\ell}} \sum_m \Psi_{\ell m} \left(\xi_m \frac{db\left(\theta\right)}{d\theta} - x_m \frac{\partial\left(\frac{\partial g}{\partial x_m}\right)}{\partial \theta}\right)}_{=\frac{dV^{\text{non-}\boxtimes}}{d\theta} \text{ (direct effect)}} + \underbrace{\sum_\ell \frac{\frac{\partial f}{\partial x_\ell}}{\frac{\partial g}{\partial x_\ell}} b\left(\theta\right) \sum_m \Psi_{\ell m} \frac{d\xi_m}{d\theta}}_{=dV^{\xi} \text{ (indirect effect)}},$$

which proves our result. □

Special Case: Linear Constraint. Whenever $g\left(\cdot\right)$ is linear, it can be written as

$$\sum_\ell g_\ell\left(\theta\right) x_\ell = b\left(\theta\right),$$

where $g_\ell\left(\theta\right)$ does not depend on any choice variable. In this case,

$$\frac{dx_\ell}{d\theta} = \frac{1}{\frac{\partial g}{\partial x_\ell}} \left(\xi_\ell \frac{db\left(\theta\right)}{d\theta} - x_\ell \frac{dg_\ell\left(\theta\right)}{d\theta} + b\left(\theta\right)\frac{d\xi_\ell}{d\theta}\right),$$

which allows us to directly write $\frac{dV}{d\theta}$ as

$$\frac{dV}{d\theta} = \underbrace{\frac{\partial f}{\partial \theta} + \sum_{\ell} \frac{\frac{\partial f}{\partial x_\ell}}{\frac{\partial g}{\partial x_\ell}} \left(\xi_\ell \frac{db\left(\theta\right)}{d\theta} - x_\ell \frac{dg_\ell\left(\theta\right)}{d\theta} \right)}_{= \frac{dV^{\text{non-}\boxtimes}}{d\theta} \text{ (direct effect)}} + \underbrace{\sum_{\ell} \frac{\frac{\partial f}{\partial x_\ell}}{\frac{\partial g}{\partial x_\ell}} b\left(\theta\right) \frac{d\xi_\ell}{d\theta}}_{= dV^{\xi} \text{ (indirect effect)}} .$$

Note that, in this case, $x_\ell = \frac{b(\theta)}{g_\ell(\theta)} \xi_\ell$, and V can be directly expressed, after a change of variables, as

$$V = \tilde{f}\left(\xi_1, \ldots \xi_\ell, \ldots, \xi_L; \theta\right) = f\left(\frac{b\left(\theta\right)}{g_1\left(\theta\right)} \xi_1, \ldots, \frac{b\left(\theta\right)}{g_\ell\left(\theta\right)} \xi_\ell, \ldots, \frac{b\left(\theta\right)}{g_L\left(\theta\right)} \xi_L; \theta\right).$$

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