ECON 500a
General Equilibrium and Welfare Economics
Static Exchange Economy

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Outline: Static Exchange Economies

1. Edgeworth Box Economy
2. Static Exchange Economy
3. Efficiency and Welfare
4. Microfounding Competition
5. Competitive Equilibrium

- Readings
  - MWG: 17.B

(General) Static Exchange Economy: $I \geq 1$, $J \geq 1$

- Pure Exchange Economy
- $I \geq 1$ individuals, indexed by $i \in \mathcal{I} = \{1, \ldots, I\}$
- $J \geq 2$ goods, indexed by $j \in \mathcal{J} = \{1, \ldots, J\}$
  - Goods are not produced, appear as endowments
- Preferences of individual i:

$$V^i = u^i \left(\left\{ c^{ij} \right\}_{j \in \mathcal{J}} \right)$$

  - Unless noted, preferences are continuous, strictly convex, and strongly monotone

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- Resource constraint of good j:

$$\sum_i c^{ij} = \bar{y}^j \quad \text{where} \quad \bar{y}^j = \sum_i \bar{y}^{ij} > 0$$

  - Ownership of endowments: $\bar{y}^{ij} \geq 0$

Static Exchange Economy: $I = 2, J = 3$

- Physical structure

$$V^1 = u^1\left(c^{11}, c^{12}, c^{13}\right) \qquad \text{Preferences Individual 1}$$
$$V^2 = u^2\left(c^{21}, c^{22}, c^{23}\right) \qquad \text{Preferences Individual 2}$$
$$c^{11} + c^{21} = \overline{y}^1 = \overline{y}^{11} + \overline{y}^{21} \qquad \text{Resource Constraint Good 1}$$
$$c^{12} + c^{22} = \overline{y}^2 = \overline{y}^{12} + \overline{y}^{22} \qquad \text{Resource Constraint Good 2}$$
$$c^{13} + c^{23} = \overline{y}^3 = \overline{y}^{13} + \overline{y}^{23} \qquad \text{Resource Constraint Good 3}$$

- Non-negative consumption: $c^{ij} \geq 0$

Notation

- Allocation

$$\mathring{\boldsymbol{c}} = \left\{ \underbrace{c^{11}, \ldots, c^{I1}}_{\text{good } 1}, \ldots, \underbrace{c^{1j}, \ldots, c^{Ij}}_{\text{good } j}, \ldots, \underbrace{c^{1J}, \ldots, c^{IJ}}_{\text{good } J} \right\}$$

  - e.g: $\boldsymbol{c}^1 = \left\{ c^{11}, \ldots, c^{1J} \right\}$
- An allocation ($\mathring{\boldsymbol{c}} \geq 0$) is *feasible* when resource constraints hold
- Endowments

$$\mathring{\bar{\boldsymbol{y}}} = \left\{ \underbrace{\bar{y}^{11}, \ldots, \bar{y}^{I1}}_{\text{good } 1}, \ldots, \underbrace{\bar{y}^{1j}, \ldots, \bar{y}^{Ij}}_{\text{good } j}, \ldots, \underbrace{\bar{y}^{1J}, \ldots, \bar{y}^{IJ}}_{\text{good } J} \right\}$$

  - e.g. $\bar{\boldsymbol{y}}^1 = \left\{ \bar{y}^{11}, \ldots, \bar{y}^{1J} \right\}$
- *Autarky* allocation is $\mathring{\boldsymbol{c}} = \mathring{\bar{\boldsymbol{y}}}$

Planning Problem

- Pareto set $\rightarrow$ allocations $\mathring{\boldsymbol{c}}$ that solve *planning problem:*

$$\max_{\mathring{\boldsymbol{c}}} \sum_i \alpha^i u^i \left(\left\{ c^{ij} \right\}_{j \in \mathcal{J}} \right)$$

  subject to

$$\sum_i c^{ij} = \bar{y}^j, \quad \forall j \in \mathcal{J},$$

  where $\sum_i \alpha^i = 1$ and $\alpha^i \geq 0$
- Varying Pareto weights α^i traces Pareto frontier

Axiomatic (Nash) Bargaining

- Contract curve → allocations $\mathring{\boldsymbol{c}}$ that solve *Nash bargaining problem*:

$$\max_{\mathring{\boldsymbol{c}}} \prod_i \left(u^i \left(\left\{ c^{ij} \right\}_{j \in \mathcal{J}} \right) - u^i \left(\left\{ \bar{y}^{ij} \right\}_{j \in \mathcal{J}} \right) \right)^{\alpha_i},$$

subject to

$$\sum_i c^{ij} = \bar{y}^j, \quad \forall j \in \mathcal{J},$$

where $\sum_i \alpha^i = 1$ and $\alpha^i \geq 0$

- Solutions to this problem satisfy
  - i) Pareto efficiency
  - ii) individual rationality
  - iii) independence of irrelevant alternatives
    If A is chosen over B in the choice set $\{A, B\}$, introducing a third option C must not result choosing B over A

Competitive/Walrasian Equilibrium

- A *competitive equilibrium* is an allocation $\mathring{\boldsymbol{c}}$ and prices $\boldsymbol{p} = \left(p^1, \ldots, p^J\right)$ such that

i) individuals choose consumption to maximize utility subject to their budget constraint taking prices as given:

$$\boldsymbol{c}^i\left(\boldsymbol{p}, \bar{\boldsymbol{y}}^i\right) = \arg\max_{\boldsymbol{c}^i} u^i\left(\left\{c^{ij}\right\}_{j\in\mathcal{J}}\right)$$

subject to

$$\sum_j p^j c^{ij} = \sum_j p^j \bar{y}^{ij}$$

ii) and markets clear, that is, resource constraints hold:

$$\sum_i c^{ij} = \sum_i \bar{y}^{ij}, \quad \forall j \in \mathcal{J}$$

Individual Problem

- Lagrangian

$$\mathcal{L} = u^i\left(\left\{c^{ij}\right\}_{j\in\mathcal{J}}\right) - \lambda^i\left(\sum_j p^j c^{ij} - \sum_j p^j \bar{y}^{ij}\right) + \nu^{ij} c^{ij}$$

- Optimality conditions given by

$$\frac{d\mathcal{L}}{dc^{ij}} = \frac{\partial u^i}{\partial c^{ij}} - \lambda^i p^j + \nu^{ij} = 0$$

- For any two goods j and $\ell \in \mathcal{J}$ consumed by i:

$$\frac{\frac{\partial u^i}{\partial c^{ij}}}{p^j} = \frac{\frac{\partial u^i}{\partial c^{i\ell}}}{p^\ell} = \lambda^i$$

- How to interpret this equation?
- In which units is each variable measured?

Excess Demand + Numeraire

- *Individual excess demand:* $z^{ij}\left(\boldsymbol{p}, \bar{\boldsymbol{y}}^i\right) = c^{ij}\left(\boldsymbol{p}, \bar{\boldsymbol{y}}^i\right) - \bar{y}^{ij}$
  - $z^{ij}\left(\boldsymbol{p}, \bar{\boldsymbol{y}}^i\right) > (<)\, 0$: i net buyer (seller) of good j
- *Aggregate excess demand*: $z^j\left(\boldsymbol{p}; \bar{\boldsymbol{y}}\right) = \sum_i z^{ij}\left(\boldsymbol{p}, \bar{\boldsymbol{y}}^i\right)$
  - $z^j\left(\boldsymbol{p}; \bar{\boldsymbol{y}}\right) > (<)\, 0$: aggregate excess demand (supply) of good j
- In vector form:

$$\boldsymbol{z}^i\left(\boldsymbol{p}, \bar{\boldsymbol{y}}^i\right) = \boldsymbol{c}^i\left(\boldsymbol{p}, \bar{\boldsymbol{y}}^i\right) - \bar{\boldsymbol{y}}^i \quad \text{and} \quad \boldsymbol{z}\left(\boldsymbol{p}; \mathring{\bar{\boldsymbol{y}}}\right) = \sum_i \boldsymbol{z}^i\left(\boldsymbol{p}, \bar{\boldsymbol{y}}^i\right)$$

- Competitive equilibrium allocations given by

$$\boxed{\boldsymbol{c}^{i\star}\left(\mathring{\bar{\boldsymbol{y}}}\right) = \boldsymbol{c}^i\left(\boldsymbol{p}^\star, \bar{\boldsymbol{y}}^i\right)} \quad \text{where} \quad \boldsymbol{p}^\star \quad \text{solves} \quad \boxed{\boldsymbol{z}\left(\boldsymbol{p}^\star; \mathring{\bar{\boldsymbol{y}}}\right) = 0}$$

- Numeraire indeterminacy $\Rightarrow$ Same as Edgeworth Box

Where Are The Diagrams?

- Only excess demand diagram is useful: when $J = 2$ and $I \geq 1$
- One could hypothetically draw
  - $I = 2$ and $J = 3 \rightarrow$ Edgeworth Cube?
  - Utility diagram with $I = 3$ and $J \geq 1$

Application: Armington Model

- Interesting case: $I = J \geq 2 \rightarrow$ Edgeworth box is $I = J = 2$
  Armington (1969); Anderson (1979)

Armington Model

- Theory of international trade based on product differentiation by country of origin
  - Goods "produced" in different countries are not perfect substitutes
  - Countries prefer to consume a variety of goods from multiple other countries rather than solely rely on the cheapest option
- Simplest model that generates a gravity equation
  Anderson and van Wincoop (2003); Arkolakis, Costinot, and Rodríguez-Clare (2012)
- Different from Ricardian model (in Block II), in which countries specialize in goods with comparative advantage in production

Application: Armington Model

- $I > 1$ individuals (countries) consume the $J = I$ goods available
- Country i is endowed with good $j = i$
  - Country 1 endowed with good 1, country 2 with good 2, ..., country I is endowed with good $J = I$
- Country i has CES preferences (likes all goods)

$\sigma \geq 0$ is elasticity of substitution, $a^{ij} > 0$ are "demand parameters"

$$V^i = \left(\sum_j \left(a^{ij}\right)^{\frac{1}{\sigma}} \left(c^{ij}\right)^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}}$$

- Budget constraint

$$\sum_j p^{ij} c^{ij} = p^i \bar{y}^i$$

- Country specific prices — p^{ij} rather than p^j — to allow for transportation costs (coming next)
- Equivalent formulation: each country has a fixed factor exclusively used to produce domestic good

Application: Armington Model

- Country i optimality conditions:

$$\frac{c^{ij}}{c^{i\ell}} = \frac{a^{ij}}{a^{i\ell}} \left(\frac{p^{ij}}{p^{i\ell}} \right)^{-\sigma}$$

- So country i's *expenditure* on good j is
  This is a version of $\boldsymbol{c}^i \left(\boldsymbol{p}, \bar{\boldsymbol{y}}^i \right)$

$$p^{ij} c^{ij} = a^{ij} \left(\frac{p^{ij}}{P^i} \right)^{1-\sigma} p^i \bar{y}^i, \quad \text{where} \quad \underbrace{P^i = \left(\sum_{\ell} \left(p^{i\ell} \right)^{1-\sigma} \right)^{\frac{1}{1-\sigma}}}_{\text{"Price Index"}}$$

- "Iceberg" shipping costs: when country j ships c^{ij} units to country i, τ^{ij} units are lost ("melt")
  This is really a technological assumption, explained in Block II
- So

$$p^{ij} = \left(1 + \tau^{ij} \right) p^j$$

Gravity Equation

- Expenditure of i on j becomes

$$p^j c^{ij} = \underbrace{a^{ij}\left(1+\tau^{ij}\right)^{-\sigma}\left(P^i\right)^{1-\sigma}}_{\text{bilateral/multilateral}} \underbrace{\left(p^j\right)^{1-\sigma}}_{j\text{- term}} \underbrace{p^i \bar{y}^i}_{i\text{- term}}$$

- This is a *generalized gravity equation* $\rightarrow$ empirical work
  - Why gravity? Flow between i and j depends on i-term, j-term, ij-term (bilateral/multilateral)
  - See e.g. Allen and Arkolakis (2016)
- First example of how one can use a GE model as a foundation for empirical work
- Note: we could close the model, but we won't

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