ECON 500a
General Equilibrium and Welfare Economics
Efficiency and Welfare

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Outline: Static Exchange Economies

1. Edgeworth Box Economy
2. Static Exchange Economy
3. $\boxed{\text{Efficiency and Welfare}}$
4. Microfounding Competition
5. Competitive Equilibrium

- Readings
  - MWG: 16.B through 16.F, 22.B, 22.C, 22.D

Outline: Efficiency and Welfare

1. $\boxed{\text{Planning Problem}}$
2. Welfare Assessments
3. First Welfare Theorem
4. Second Welfare Theorem
5. Application: Transfer Problem/Paradox
6. Application: Optimal Taxation

Planning Problem

- What are the properties of Pareto efficient allocations?
  - Let's solve the planning problem
- Two methods
  1. Lagrangian
  2. Perturbation
- Planning Problem:

$$\max_{\mathring{\boldsymbol{c}}} \sum_i \alpha^i u^i \left(\left\{ c^{ij} \right\}_{j \in \mathcal{J}} \right)$$

subject to

$$\sum_i c^{ij} = \bar{y}^j, \quad \forall j \in \mathcal{J}$$

Planning Problem: Lagrangian

- Lagrangian

$$\mathcal{L} = \sum_i \alpha^i u^i \left(\left\{ c^{ij} \right\}_{j \in \mathcal{J}} \right) - \sum_j \eta^j \left(\sum_i c^{ij} - \bar{y}^j \right) + \sum_i \sum_j \kappa^{ij} c^{ij}$$

  - $\eta^j \geq 0$: multiplier in resource constraint (J constraints)
  - $\kappa^{ij} \geq 0$: multiplier in non-negativity constraints (IJ constraints)

- Optimality conditions:

$$\frac{d\mathcal{L}}{dc^{ij}} = \alpha^i \frac{\partial u^i}{\partial c^{ij}} - \eta^j + \kappa^{ij} = 0$$

  - $\alpha^i \frac{\partial u^i}{\partial c^{ij}}$: Marginal Benefit of increasing c^{ij}
  - η^j: Marginal Cost of increasing c^{ij}

- If $c^{ij} = 0 \to \kappa^{ij} = \eta^j - \alpha^i \frac{\partial u^i}{\partial c^{ij}} > 0 \to \eta^j > \alpha^i \frac{\partial u^i}{\partial c^{ij}}$
  - Marginal Cost > Marginal Benefit

Planning Problem: Lagrangian

- Let's focus on interior allocations
- Combine optimality conditions for any two goods (say j and $\ell \in \mathcal{J}$) consumed by i (also $n \in \mathcal{I}$)

$$\frac{\frac{\partial u^i}{\partial c^{ij}}}{\frac{\partial u^i}{\partial c^{i\ell}}} = \frac{\eta^j}{\eta^\ell} \Rightarrow \boxed{\frac{\frac{\partial u^i}{\partial c^{ij}}}{\frac{\partial u^i}{\partial c^{i\ell}}} = \frac{\frac{\partial u^n}{\partial c^{nj}}}{\frac{\partial u^n}{\partial c^{n\ell}}} = \frac{\eta^j}{\eta^\ell}} \quad \Rightarrow \quad \text{Efficiency}$$

  - $\frac{\frac{\partial u^i}{\partial c^{ij}}}{\frac{\partial u^i}{\partial c^{i\ell}}}$ is i's MRS between j and ℓ
  - MRS's equalized across individuals $\rightarrow$ equal to ratio of shadow prices
- Edgeworth box: tangent indifference curves
- The boxed condition is <u>invariant</u> to preference-preserving transformations and Pareto weights

Planning Problem: Lagrangian

- Combine optimality conditions for any two individuals i and n who consume good j (or good ℓ)

$$\frac{\frac{\partial u^i}{\partial c^{ij}}}{\frac{\partial u^n}{\partial c^{nj}}} = \frac{\alpha^n}{\alpha^i} \Rightarrow \boxed{\frac{\frac{\partial u^i}{\partial c^{ij}}}{\frac{\partial u^n}{\partial c^{nj}}} = \frac{\frac{\partial u^i}{\partial c^{i\ell}}}{\frac{\partial u^n}{\partial c^{n\ell}}} = \frac{\alpha^n}{\alpha^i}} \quad \Rightarrow \quad \text{Redistribution}$$

  - Ratio of marginal utilities equals to ratio of Pareto weights $\frac{\alpha^n}{\alpha^i}$
  - This condition encodes planner's preferences
  - Intuition: $\alpha^i \uparrow \implies \frac{\partial u^i}{\partial c^{ij}} \downarrow \implies c^{ij} \uparrow$
- Edgeworth Box: this condition picks a precise point in the Pareto frontier in the utility diagram
- The boxed condition is <u>not invariant</u> to preference-preserving transformations and Pareto weights

Planning Problem: Perturbation

- Social Welfare
$$W = \sum_i \alpha^i V^i$$
  - Units of W: Social utils
  - Units of V^i: individual i utils
- Perturbation: indexed by θ (θ is a placeholder)
  - We could have used differentials and write dW instead of $\frac{dW}{d\theta}$
- How do we find an optimum?
  - Allocations for which all feasible perturbations satisfy $\frac{dW}{d\theta} \leq 0$
  - At interior allocations: $\frac{dW}{d\theta} = 0$

Planning Problem: Perturbation

- Welfare change

$$\frac{dW}{d\theta} = \sum_i \alpha^i \frac{dV^i}{d\theta}, \quad \text{where} \quad \frac{dV^i}{d\theta} = \sum_j \frac{\partial u^i}{\partial c^{ij}} \frac{dc^{ij}}{d\theta}$$

- Therefore

$$\frac{dW}{d\theta} = \sum_i \sum_j \alpha^i \frac{\partial u^i}{\partial c^{ij}} \frac{dc^{ij}}{d\theta}$$

- But feasible perturbations satisfy

$$\sum_i \frac{dc^{ij}}{d\theta} = \frac{d\bar{y}^j}{d\theta} = 0$$

- What is the condition for efficiency?
  - $\alpha^i \frac{\partial u^i}{\partial c^{ij}}$ must be equal for all $i \rightarrow \boxed{\alpha^i \frac{\partial u^i}{\partial c^{ij}} = \alpha^n \frac{\partial u^n}{\partial c^{nj}}}$
  - Otherwise, shift one unit of consumption of good j from individual i to n (or viceversa), with social gain: $\frac{dc^{ij}}{d\theta} = 1$ and $\frac{dc^{nj}}{d\theta} = -1$

$$\frac{dW}{d\theta} = \alpha^i \frac{\partial u^i}{\partial c^{ij}} - \alpha^n \frac{\partial u^n}{\partial c^{nj}} > 0$$

Planning Problem: Perturbation

- But $\alpha^i \frac{\partial u^i}{\partial c^{ij}} = \alpha^n \frac{\partial u^n}{\partial c^{nj}}$ implies

$$\frac{\frac{\partial u^i}{\partial c^{ij}}}{\frac{\partial u^i}{\partial c^{i\ell}}} = \frac{\frac{\partial u^n}{\partial c^{nj}}}{\frac{\partial u^n}{\partial c^{n\ell}}} = \frac{\eta^j}{\eta^\ell} \quad \text{and} \quad \frac{\frac{\partial u^i}{\partial c^{ij}}}{\frac{\partial u^n}{\partial c^{nj}}} = \frac{\frac{\partial u^i}{\partial c^{i\ell}}}{\frac{\partial u^n}{\partial c^{n\ell}}} = \frac{\alpha^n}{\alpha^i}$$

- Perturbation approach $\rightarrow$ mathematical foundation of Lagrangian
  Sadly, Lagrangian approach is typically taught as a cookbook :(
- Planning approach useful because it yields Lagrange multipliers
- Perturbation approach useful for welfare assessments (next)

Remarks on Pareto Efficiency/Optimality

i) Pareto optimality is mute about distributional consequences
   - NE and SW allocations in Edgeworth box are Pareto efficient → one agent is very unhappy

ii) Economists have little to say about which Pareto efficient allocation is preferred
   - Social Welfare Functions (coming next) are our tool to capture social preferences for redistribution
   - Normative statements that involve interpersonal comparisons must be caveated by the choice of SWF and utility units

iii) Pareto criterion defines an incomplete order
   - There are always winners and losers – especially as I grows
   - Alternative: Kaldor-Hicks potential compensation principle

iv) Pareto efficiency of an allocation depends on what is feasible
   Pareto efficiency vs. Constrained Pareto efficiency
   - Pareto efficiency for us (unless noted) → allocation solves planning problem ("resource-feasible")
     Feasible perturbations defined by resource constraints
   - Later in the course → *constrained* Pareto efficiency
     - Set of feasible perturbations is constrained

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Welfare Assessments

- Many questions take the form of "welfare assessments"?
  1. Policy counterfactuals: welfare impact of a tax change?
  2. Change in primitives: welfare impact of a change in technology?
  3. Optimal policy exercises: what is the optimal tax system?
- Given a physical structure, can we systematically attribute the welfare gains of a perturbation to specific sources? Yes!
  - Question of $\boxed{\text{"}\textit{Origins of welfare gains}\text{"}}$

    Dávila and Schaab (2024) and Dávila and Schaab (2024b) and more
- Say we compute $\frac{dW}{d\theta} = 0.4$ for a perturbation

  e.g. policy, change in allocation, endowments, etc.
  - Why did we get this number?
  - How do we interpret $\frac{dW}{d\theta} > 0$?
  - And $\frac{dW}{d\theta} = 0.4$?

Welfarism and Social Welfare Functions (SWF)

- Welfarist planners assess welfare via a *social welfare function*:
  Bergson (1938), Samuelson (1947)

$$W = \mathcal{W}\left(V^1, \ldots, V^i, \ldots, V^I\right) \qquad \text{(Social Welfare Function)}$$

  - W induces "social" ranking over allocations $\mathring{\boldsymbol{c}}$: $W = \mathcal{W}\left(\{\mathring{\boldsymbol{c}}\}\right)$
    Assumption: $\frac{\partial \mathcal{W}}{\partial V^i} > 0, \forall i$ (everybody counts) $\rightarrow$ can be relaxed
- A welfarist planner finds a perturbation desirable (undesirable) if

$$\frac{dW}{d\theta} = \sum_i \frac{\partial \mathcal{W}}{\partial V^i} \frac{dV^i}{d\theta} > (<)\, 0$$

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$$\frac{dW}{d\theta} = \sum_i \frac{\partial \mathcal{W}}{\partial V^i} \frac{dV^i}{d\theta} > (<)\, 0$$

- Critical restriction of $\boxed{\text{welfarism}}$: SWF $\mathcal{W}(\cdot)$ exclusively depends on individual utility levels V^i (Kaplow, 2011)
  - Welfarist approach is Paretian: every Pareto improving perturbation is desirable
    - Proof: If $\frac{dV^i}{d\theta} \geq 0$, $\forall i$, with one strict inequality, then $\frac{dW}{d\theta} > 0$
  - Converse statement true under minimal assumptions
    Kaplow and Shavell (2001): "Any Non-Welfarist Method of Policy Assessment Violates the Pareto Principle"

Welfarism and Social Welfare Functions (SWF)

- Most used SWF:

$$\text{Utilitarian :} \quad W = \sum_i \alpha^i V^i \quad \text{or} \quad W = \sum_i V^i$$

  - Linearity is useful because it traces (convex) Pareto frontier
  - There are other SWF: CES, Cobb-Douglas, Rawlsian, etc.
    Kaplow (2011), MWG
- Problem with SWF → Utilities are ordinal (!!)
  - "It's the <u>units</u>, stupid"

Addressing the Units Issue

- Let's choose a unit to make interpersonal comparisons

$$\boxed{\text{Welfare numeraire}}$$

- Welfare change:

$$\frac{dW}{d\theta} = \sum_i \frac{\partial \mathcal{W}}{\partial V^i} \frac{dV^i}{d\theta} = \sum_i \frac{\partial \mathcal{W}}{\partial V^i} \lambda^i \frac{\frac{dV^i}{d\theta}}{\lambda^i} = \sum_i \frac{\partial \mathcal{W}}{\partial V^i} \lambda^i \frac{dV^{i|\lambda}}{d\theta}$$

  - $\lambda^i > 0$ is an individual normalizing factor

$$\dim\left(\lambda^i\right) = \frac{\text{utils of individual } i}{\text{units of welfare numeraire}}$$

- Units of $\frac{dV^{i|\lambda}}{d\theta} = \frac{\frac{dV^i}{d\theta}}{\lambda^i}$ are common $\forall i \rightarrow$ This is great!

$$\dim\left(\frac{\frac{dV^i}{d\theta}}{\lambda^i}\right) = \frac{\frac{\text{utils of individual } i}{\text{units of } \theta}}{\frac{\text{utils of individual } i}{\text{units of welfare numeraire}}} = \frac{\text{units of welfare numeraire}}{\text{units of } \theta}$$

- $\frac{dV^{i|\lambda}}{d\theta}$ corresponds to individuals i's "willingness to pay" for perturbation in units of the (common) welfare numeraire
  - We have "cardinalized" $\frac{dV^i}{d\theta}$

Addressing the Units Issue

- But what is exactly λ^i?
  - $\lambda^i = \frac{\partial u^i}{\partial c^{i1}}$: 1 unit of good 1 as welfare numeraire
  - $\lambda^i = \frac{\partial u^i}{\partial c^{i2}}$: 1 unit of good 2 as welfare numeraire
  - $\lambda^i = \frac{\partial u^i}{\partial c^{i1}} + \frac{\partial u^i}{\partial c^{i2}}$: one unit of bundle of $\{$1 unit of good 1 and 1 unit of good 2$\}$ as welfare numeraire
  - $\lambda^i = \frac{\partial u^i}{\partial c^{i1}} c^1$ with $c^1 = \sum_i c^{i1}$: 1 proportional unit of aggregate good 1's consumption as welfare numeraire
  - $\lambda^i = \frac{\partial u^i}{\partial c^{i1}} c^1 + \frac{\partial u^i}{\partial c^{i2}} c^2$: proportional bundle

Cardinalizing Social Welfare

- Let's also cardinalize social welfare:
  Normalize $\frac{dW}{d\theta}$ by $\frac{1}{I}\sum_i \frac{\partial \mathcal{W}}{\partial V^i}\lambda^i$

$$\frac{dW^\lambda}{d\theta} = \frac{\frac{dW}{d\theta}}{\frac{1}{I}\sum_i \frac{\partial \mathcal{W}}{\partial V^i}\lambda^i} = \sum_i \omega^i \frac{dV^{i|\lambda}}{d\theta} \quad \text{where} \quad \omega^i = \frac{\frac{\partial \mathcal{W}}{\partial V^i}\lambda^i}{\frac{1}{I}\sum_i \frac{\partial \mathcal{W}}{\partial V^i}\lambda^i}$$

  - $\frac{dW^\lambda}{d\theta}$ in units of welfare numeraire equally distributed (with $\frac{1}{I}$ shares) to all individuals
- ω^i are "normalized individual weights"
  - Average to one $\frac{1}{I}\sum_i \omega^i = 1$
  - If $\omega^1 = 1.1$, a planner values the same giving 1 unit of welfare numeraire to $i = 1$ and 1.1 units of welfare numeraire equally distributed to everyone.
  - If and $\omega^2 = 0.9 \Rightarrow \frac{\omega^1}{\omega^2} = \frac{1.1}{0.9} = 1.22$: we can say that planner likes individual $i = 1$ 22% more than $i = 2$.
- Note that ω^i depends on
  - SWF
  - Utility units

Efficiency/Redistribution Decomposition

- A normalized welfare assessment is a weighted sum of normalized individual welfare gains:

$$\frac{dW^{\lambda}}{d\theta} = \sum_i \omega^i \frac{dV^{i|\lambda}}{d\theta}$$

- Efficiency/Redistribution Decomposition:

$$\frac{dW^{\lambda}}{d\theta} = \underbrace{\sum_i \frac{dV^{i|\lambda}}{d\theta}}_{\Xi^E \text{ (Efficiency)}} + \underbrace{\mathbb{C}ov_i^{\Sigma}\left[\omega^i, \frac{dV^{i|\lambda}}{d\theta}\right]}_{\Xi^{RD} \text{ (Redistribution)}}$$

- Cross-sectional covariance decomposition:

$$\sum_i x^i y^i = \frac{1}{I}\sum_i x^i \sum_i y^i + \underbrace{I\mathbb{C}ov_i\left[x^i, y^i\right]}_{\equiv \mathbb{C}ov_i^{\Sigma}[\cdot,\cdot]}$$

Efficiency/Redistribution Decomposition

$$\boxed{\frac{dW^{\lambda}}{d\theta} = \underbrace{\sum_i \frac{dV^{i|\lambda}}{d\theta}}_{\Xi^E \text{ (Efficiency)}} + \underbrace{\mathbb{C}ov_i^{\Sigma}\left[\omega^i, \frac{dV^{i|\lambda}}{d\theta}\right]}_{\Xi^{RD} \text{ (Redistribution)}}}$$

- Efficiency is *Kaldor-Hicks efficiency*
  - Sum of individual willingness-to-pay
- Kaldor-Hicks efficiency based on *compensation principle*:
  (Kaldor, 1939; Hicks, 1939; Boadway and Bruce, 1984)

  "a perturbation is desirable if the winners can <u>hypothetically</u> compensate the losers."

Pareto vs. Kaldor-Hicks Efficiency

i) Perturbations with $\Xi^E > 0$ can be turned into Pareto improvements if transfers were feasible and costless

ii) If a perturbation is Pareto improving, then $\Xi^E > 0$

- Since Pareto-improving perturbations have no losers $\rightarrow$ sum of willingness-to-pay must be strictly positive

iii) (Interior) Pareto efficient allocations must have $\Xi^E \leq 0$ for any feasible perturbation

- Otherwise do the perturbation and redistribute the gains!
- Not true for perturbations of *constrained* Pareto efficient allocations

Properties

$$\frac{dW^{\lambda}}{d\theta} = \underbrace{\sum_i \frac{dV^{i|\lambda}}{d\theta}}_{\Xi^E \text{ (Efficiency)}} + \underbrace{\mathbb{C}ov_i^{\Sigma}\left[\omega^i, \frac{dV^{i|\lambda}}{d\theta}\right]}_{\Xi^{RD} \text{ (Redistribution)}}$$

$$\boxed{\text{Welfarism} = \text{Kaldor-Hicks} + \text{Redistribution}}$$

- Efficiency component is invariant to
  - i) the choice of social welfare function
  - ii) preference-preserving utility transformations.
- Redistribution component can have any sign
  - It is possible to select individual weights ω^i (by varying the SWF or individual utility units) so that Ξ^{RD} is positive or negative for a given perturbation
  - Ξ^{RD} can be negative for Pareto-improving perturbations, even though $\Xi^E + \Xi^{RD} > 0$.

Welfare Accounting

- Up to now $\rightarrow$ We only assumed differentiability!
- Back to static exchange economy
- From planning perspective $\rightarrow$ only two possible perturbations
  - Consumption allocations $\rightarrow \frac{dc^{ij}}{d\theta} \gtreqless 0$
  - Endowments $\rightarrow \frac{d\bar{y}^{j}(\theta)}{d\theta} \gtreqless 0$
- Therefore (remember that $\frac{dV^{i}}{d\theta} = \sum_{j} \frac{\partial u^{i}}{\partial c^{ij}} \frac{dc^{ij}}{d\theta}$)

$$\frac{dV^{i|\lambda}}{d\theta} = \sum_{j} \underbrace{\frac{\frac{\partial u^{i}}{\partial c^{ij}}}{\lambda^{i}}}_{=MRS_c^{ij}} \frac{dc^{ij}}{d\theta} = \sum_{j} MRS_c^{ij} \frac{dc^{ij}}{d\theta}$$

- Definitions
  i) individual i's consumption share of good j is $\chi_c^{ij} = \frac{c^{ij}}{c^{j}}$
  ii) aggregate consumption of good j is $c^{j} = \sum_{i} c^{ij}$
- Hence

$$\boxed{\frac{dc^{ij}}{d\theta} = \frac{d\chi_c^{ij}}{d\theta} c^{j} + \chi_c^{ij} \frac{dc^{j}}{d\theta}}$$

Welfare Accounting

- Putting all together

$$\Xi^E = \sum_i \frac{dV^{i|\lambda}}{d\theta} = \sum_i \sum_j MRS_c^{ij} \frac{dc^{ij}}{d\theta}$$
$$= \sum_j \sum_i MRS_c^{ij} \frac{d\chi_c^{ij}}{d\theta} c^j + \sum_j \underbrace{\sum_i \chi_c^{ij} MRS_c^{ij}}_{=AMRS_c^j} \frac{dc^j}{d\theta}$$
$$= \sum_j \mathbb{C}ov_i^{\Sigma}\left[MRS_c^{ij}, \frac{d\chi_c^{ij}}{d\theta}\right] c^j + \sum_j AMRS_c^j \frac{dc^j}{d\theta}$$

- $AMRS$: aggregate marginal rate of substitution
  - $AMRS_c^j$ corresponds to the marginal social value of having an extra unit of good j (MSV^j) if we do nothing
  - "Doing nothing" $\rightarrow$ Allocation shares $\frac{d\chi_c^{ij}}{d\theta}$ remain fixed
    Dávila and Schaab (2024a): "Non-Envelope Theorem"
- Recourse constraint implies $\frac{dc^j}{d\theta} = \frac{d\bar{y}^j(\theta)}{d\theta}$

Welfare Accounting

$$\boxed{\Xi^{E} = \underbrace{\sum_{j} \mathbb{C}ov_{i}^{\Sigma}\left[MRS_{c}^{ij}, \frac{d\chi_{c}^{ij}}{d\theta}\right] c^{j}}_{\substack{\text{Cross-Sectional Consumption} \\ \text{Exchange Efficiency}}} + \underbrace{\sum_{j} AMRS_{c}^{j} \frac{d\bar{y}^{j}(\theta)}{d\theta}}_{\text{Good Endowment Change}}}$$

- What are the origins of welfare (efficiency) gains?
- Efficiency gains in static exchange economies are due to either
  - ii) Exchange efficiency (reallocating consumption to individuals with higher MRS_{c}^{ij})
  - ii) Endowment change (having more goods to consume)
- For $\boxed{\text{any economic structure!}}$
  - This equation only uses preferences + resource constraints

Welfare Accounting $\rightarrow$ Efficiency?

$$\boxed{\Xi^E = \underbrace{\sum_j \mathbb{C}ov_i^{\Sigma}\left[MRS_c^{ij}, \frac{d\chi_c^{ij}}{d\theta}\right] c^j}_{\substack{\text{Cross-Sectional Consumption} \\ \text{Exchange Efficiency}}} + \underbrace{\sum_j AMRS_c^j \frac{d\bar{y}^j(\theta)}{d\theta}}_{\text{Good Endowment Change}}}$$

- Which condition makes the first-term zero for any perturbation $\frac{d\chi_c^{ij}}{d\theta}$?
  - $MRS_c^{ij} = MRS_c^{nj}, \forall i, n \in \mathcal{I}, \forall j \in \mathcal{J}$
  - Same as the perturbation approach to planning
- What if we are at an efficient allocation?
  - Remember $MRS_c^{ij} = \frac{\frac{\partial u^i}{\partial c^{ij}}}{\lambda^i}$ and take $\lambda^i = \frac{\partial u^i}{\partial c^{i\ell}}$ (ℓ can be any good)
  - $MRS_c^{ij} = \frac{\frac{\partial u^i}{\partial c^{ij}}}{\frac{\partial u^i}{\partial c^{i\ell}}} = \frac{\frac{\partial u^n}{\partial c^{nj}}}{\frac{\partial u^n}{\partial c^{n\ell}}} = MRS_c^{nj} \Rightarrow$ Exchange Efficiency term is zero!

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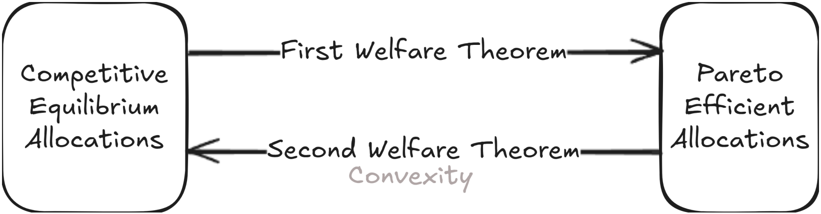
Welfare Theorems

- What is the relation between competition and efficiency?



- First Welfare Theorem: every competitive equilibrium allocation is Pareto efficient
  - So important that we will see *three* different proofs!
- Second Welfare Theorem: every Pareto efficient allocation can be decentralized as a competitive equilibrium (with transfers)

Three Proofs of First Welfare Theorem

Proof #1 Arrow (1951) and Debreu (1951) (independently)

- Minimal assumptions $\rightarrow$ most powerful
- Only requires non-satiation

Proof #2 Lange (1942)

- Highlights connection with planning problem
- Joint proof of first and second welfare theorems
- Useful for computation $\Rightarrow$ Negishi approach

Proof #3 Adapted from Geanakoplos and Polemarchakis (1986) and Dávila and Korinek (2018)

- Highlights role of pecuniary effects (prices)
- Useful to proof constrained inefficiency with incomplete markets $\rightarrow$ Block III

- Downsides
  - Proof #1: lack of economic mechanisms
  - Proof #2: requires differentiability + no pecuniary effects
  - Proof #3: requires differentiability + local argument

Proof #1 of First Welfare Theorem

- Consider CE $\rightarrow \mathring{\boldsymbol{c}}^\star$ and $\boldsymbol{p}^\star$
- Suppose another feasible allocation $\mathring{\boldsymbol{c}}$ Pareto dominates $\mathring{\boldsymbol{c}}^\star$, with:
  i) At least one individual strictly better off: $V^i > V^{i\star}$ for some i.
  ii) No individual worse off: $V^i \geq V^{i\star}$, $\forall i$.
- The strictly better off individual could not have afforded the new allocation at competitive prices, so

$$\sum_j p^{j\star} c^{ij} > \sum_j p^{j\star} \bar{y}^{ij}$$

- Local non-satiation ensures that, for all other individuals:

$$\sum_j p^{j\star} c^{ij} \geq \sum_j p^{j\star} \bar{y}^{ij}$$

- Aggregating

$$\sum_i \sum_j p^{j\star} c^{ij} > \sum_i \sum_j p^{j\star} \bar{y}^{ij} \Rightarrow \sum_j p^{j\star} \left(\sum_i c^{ij} - \sum_i \bar{y}^{ij} \right) > 0$$

- But market clearing requires $\sum_i c^{ij} = \sum_i \bar{y}^{ij}$, which contradicts the previous equation
  - Hence, no feasible allocation $\mathring{\boldsymbol{c}}$ Pareto dominates $\mathring{\boldsymbol{c}}^\star$
  - Any competitive equilibrium is Pareto efficient

Proof #2 of First Welfare Theorem

- Consider interior case with $c^{ij} > 0$ (can be relaxed)
- Individual optimality conditions
  λ^i: Lagrange multiplier on budget constraint
$$\frac{\partial u^i}{\partial c^{ij}} - \lambda^i p^j = 0$$
- Planning optimality conditions
  α^i: Pareto weight and η^j: good j's Lagrange multiplier
$$\frac{\partial u^i}{\partial c^{ij}} - \frac{1}{\alpha^i}\eta^j = 0$$
- Hence, there are one-to-one mappings between λ^i and α^i, and between η^j and p^j:
$$\lambda^i \leftrightarrow \frac{1}{\alpha^i} \quad \text{and} \quad p^j \leftrightarrow \eta^j.$$
- Given a CE, if we choose Pareto weights $\alpha^i = \frac{1}{\lambda^i}$, we know that $p^j = \eta^j$ is a solution of the planning problem
  - Any competitive equilibrium is Pareto efficient

Proof #3 of First Welfare Theorem

- Starting from a CE, compute individual welfare gains of a perturbation:
  λ^i: Lagrange multiplier on budget constraint

$$\frac{dV^i}{d\theta} = \sum_j \frac{\partial u^i}{\partial c^{ij}} \frac{dc^{ij}}{d\theta} = \lambda^i \sum_j \frac{\frac{\partial u^i}{\partial c^{ij}}}{\lambda^i} \frac{dc^{ij}}{d\theta} = \lambda^i \sum_j p^j \frac{dc^{ij}}{d\theta}$$

  - Last equation uses individual optimality
- Say we perturb individual demands, but budget constraints and market clearing remain satisfied:

$$\sum_j p^j \frac{dc^{ij}}{d\theta} + \sum_j \frac{dp^j}{d\theta} c^{ij} = \sum_j \frac{dp^j}{d\theta} \bar{y}^{ij} \implies \sum_j p^j \frac{dc^{ij}}{d\theta} = \underbrace{\sum_j \frac{dp^j}{d\theta} \left(\bar{y}^{ij} - c^{ij}\right)}_{\text{Distributive Pecuniary Effects}}$$

- Changes in value of individual i's consumption, $\sum_j p^j \frac{dc^{ij}}{d\theta}$, equals *distribute pecuniary effects,* composed of
  Language as in Dávila and Korinek (2018)
  i) net trading positions (net buying/selling): $\bar{y}^{ij} - c^{ij}$
  ii) sensitivity of equilibrium prices: $\frac{dp^j}{d\theta}$

Proof #3 of First Welfare Theorem (cont.)

- Normalized individual welfare gain $\frac{\frac{dV^i}{d\theta}}{\lambda^i}$ is

$$\frac{\frac{dV^i}{d\theta}}{\lambda^i} = \sum_j p^j \frac{dc^{ij}}{d\theta} = \sum_j dp^j \left(\bar{y}^{ij} - c^{ij}\right)$$

- Aggregating across all i:

$$\sum_i \frac{\frac{dV^i}{d\theta}}{\lambda^i} = \sum_j \frac{dp^j}{d\theta} \underbrace{\sum_i \left(\bar{y}^{ij} - c^{ij}\right)}_{=0} = 0$$

- Last equality follows from market clearing
    - Distributive pecuniary effects *"cancel out"* or *"add up to zero"* (not true with incomplete markets!)
- Final step by contradiction
    - Since $\sum_i \frac{\frac{dV^i}{d\theta}}{\lambda^i} = 0$, if $\frac{\frac{dV^i}{d\theta}}{\lambda^i} > 0$ for someone, there must be another individual for whom $\frac{\frac{dV^i}{d\theta}}{\lambda^i} < 0$
    - Any competitive equilibrium is Pareto efficient

Proof #3 of First Welfare Theorem (cont.)

- Agents exclusively interact through prices $\rightarrow$ spillovers/externalities among individuals must operate via prices
- If a perturbation increases (decreases) the prices of the goods that individual i purchases (sales), i will be worse off, and vice versa
  - These are *pecuniary* externalities
- Sum of pecuniary externalities/effects is zero
  - There are always winners and losers
  - No possible Pareto improvements

Assumptions First Welfare Theorem

- Several assumptions are critical for first welfare theorem:
  i) local non-satiation, as highlighted by the first proof
  ii) price-taking behavior or, equivalently, absence of market power
  iii) self-interested preferences or, equivalently, absence of consumption externalities
  iv) markets for each of the goods, or, equivalently, complete markets for goods
- In production economies:
  - No public goods
  - No production externalities
- In dynamic stochastic economies:
  - Incomplete markets
  - Asymmetric information
  - Belief distortions
  - Double infinity
- Continuity, differentiability, or convexity of preferences not required for first welfare theorem
  - They may be critical assumptions to ensure existence of a competitive equilibrium

Formalizing the Invisible Hand

- $\boxed{\text{First Welfare Theorem}}$ → individuals pursuing their self-interest in a competitive market will, without intending it, achieve an allocation of resources that cannot be (Pareto) improved upon
- Formalizing Adam Smith's concept of the invisible hand:

  "Every individual (...) neither intends to promote the public interest, nor knows how much he is promoting it. By directing that industry in such a manner as its produce may be of the greatest value, he intends only his own gain, and he is (...) led by an invisible hand to promote an end which was no part of his intention."

  The Wealth of Nations, Book IV, Chapter II.

- Paradoxical reliance on the assumption of individual self-interest:

  "It is not from the benevolence of the butcher, the brewer, or the baker that we expect our dinner, but from their regard to their own interest."

  The Wealth of Nations, Book I, Chapter II.

Outline: Efficiency and Welfare

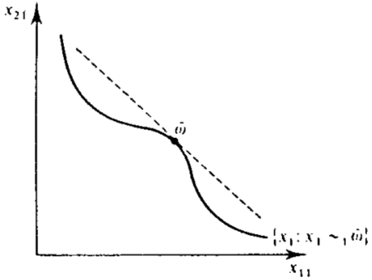
1. Planning Problem
2. Welfare Assessments
3. First Welfare Theorem
4. $\boxed{\text{Second Welfare Theorem}}$
5. Application: Transfer Problem/Paradox
6. Application: Optimal Taxation

Second Welfare Theorem

- Second Welfare Theorem: every Pareto efficient allocation can be decentralized as a competitive equilibrium (with transfers)
  - Second proof of first welfare theorem is a constructive proof of second welfare theorem
  - General proof applies Hahn–Banach separation theorem (separating hyperplane theorem) to feasible and desired allocations

    My favorite proof is in Stokey, Lucas, and Prescott (1989)
- Proof is easy in Edgeworth box
  - Allocation $\rightarrow$ slope $\rightarrow$ prices $\rightarrow$ endowments

Assumptions Second Welfare Theorem



- Second welfare theorem requires <u>convexity</u> (also non-satiation)
  - As $I \to \infty$, convexity less important (coming next)
- More important in practice: planners <u>cannot</u> transfer resources costlessly!
  - $\boxed{\text{Optimal taxation}}$

Outline: Efficiency and Welfare

Application: Transfer Problem/Paradox

- *Transfer problem:* how does the transfer of resources from one country to another affects allocations, prices, and welfare?
  - Motivated by German reparations post-WWI
  - Keynes (1929) vs. Ohlin (1929)
- *Transfer paradox:* can the transfer of resources from one country to another lead to a worsening of the recipient's welfare and an improvement of the donor's welfare?

$\boxed{\text{We already have the tools to tackles these issues!}}$

Application: Transfer Problem/Paradox

- Consider perturbation of individual endowments — as usual indexed by θ — in a competitive environment
- i's budget constraint implies that

$$\sum_j p^j \frac{dc^{ij}}{d\theta} + \sum_j \frac{dp^j}{d\theta} \left(c^{ij} - \bar{y}^{ij}\right) - \sum_j p^j \frac{d\bar{y}^{ij}}{d\theta} = 0$$

- Individual welfare gains are

$$\frac{\frac{dV^i}{d\theta}}{\lambda^i} = \sum_j p^j \frac{dc^{ij}}{d\theta} = \underbrace{\sum_j \frac{dp^j}{d\theta} \left(\bar{y}^{ij} - c^{ij}\right)}_{\substack{\text{Distributive Pecuniary} \\ \text{Effects}}} + \underbrace{\sum_j p^j \frac{d\bar{y}^{ij}}{d\theta}}_{\substack{\text{Direct Transfer} \\ \text{Effects}}}$$

- Direct transfer effects easy to sign
  - If i receives a transfer of good j $\rightarrow$ welfare increase in proportion to p^j
  - If i transfers away good j $\rightarrow$ welfare loss in proportion to p^j
- Distributive pecuniary effects benefit:
  - Net sellers of good j, $\bar{y}^{ij} - c^{ij} > 0$, if j's price increases, $\frac{dp^j}{d\theta} > 0$
  - Net buyers of good j, $\bar{y}^{ij} - c^{ij} < 0$, if j's price decreases, $\frac{dp^j}{d\theta} < 0$

Application: Transfer Problem/Paradox

- Both distributive pecuniary effects and direct transfers effects cancel out in the aggregate:

$$\sum_i \sum_j \frac{dp^j}{d\theta}\left(\bar{y}^{ij} - c^{ij}\right) = 0 \quad \text{and} \quad \sum_i \sum_j p^j \frac{d\bar{y}^{ij}}{d\theta} = 0$$

  - Because of market clearing: $\sum_i \left(\bar{y}^{ij} - c^{ij}\right) = 0$
  - Transfer is zero-sum nature: $\sum_i \frac{d\bar{y}^{ij}}{d\theta} = 0$
- Therefore, welfare gains also cancel out: $\sum_i \frac{\frac{dV^i}{d\theta}}{\lambda^i} = 0$
  - But transfer paradox can exist! When?
  - Country with $\frac{d\bar{y}^{ij}}{d\theta} < 0$ experiences large beneficial distributive pecuniary effects
  - When? The recipient country exerts a lot of upward pressure on the goods of the donor country starting from a situation in which the donor's exports are large

Application: Transfer Problem/Paradox

- Leontief (1936) shows transfer paradox can happen when $I = 2$
- Samuelson (1952) shows transfer paradox can only occur for unstable equilibrium when $I = 2$
- Transfer paradox can occur at unique and stable equilibria when $I > 3$
  - Polemarchakis (1983) provides fully worked out application along the lines of the exposition of this section
  - See also Chichilnisky (1980) and Geanakoplos and Heal (1983)

Outline: Efficiency and Welfare

1. Planning Problem
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6. $\boxed{\text{Application: Optimal Taxation}}$

Application: Optimal Taxation

- Second welfare theorem assumes *"lump-sum" taxes/transfers*
  - In practice $\rightarrow$ taxes need to be conditioned on economic acts, e.g. consumption
- How to achieve a particular social objective (e.g. redistribution), minimizing efficiency losses imposed?
  - Public Economics/Finance $\Rightarrow$ $\boxed{\text{Theory of Optimal Taxation}}$
- In static exchange economies $\rightarrow$ consumption taxes
- Budget constraint of i is

$$\sum_j \left(1+\tau^{ij}\right) p^j c^{ij} = \sum_j p^j \bar{y}^{ij} + T^i \Rightarrow \sum_j p^j c^{ij} = \sum_j p^j \bar{y}^{ij} + \underbrace{\overbrace{T^i}^{\text{Rebate}} - \overbrace{\sum_j \tau^{ij} p^j c^{ij}}^{\text{Tax Payment}}}_{\text{Net Transfer}}$$

  - T^i taken as given
- Net transfers add up to zero in the aggregate:

$$\sum_i T^i = \sum_i \sum_j \tau^{ij} p^j c^{ij}$$

Application: Optimal Taxation

- Optimality conditions:

$$\frac{\partial u^i}{\partial c^{ij}} = \lambda^i \left(1 + \tau^{ij}\right) p^j$$

- Individual i's welfare gains

$$\frac{dV^i}{d\theta} = \sum_j \frac{\partial u^i}{\partial c^{ij}} \frac{dc^{ij}}{d\theta} = \lambda^i \sum_j \frac{\frac{\partial u^i}{\partial c^{ij}}}{\lambda^i} \frac{dc^{ij}}{d\theta} = \lambda^i \sum_j \left(1 + \tau^{ij}\right) p^j \frac{dc^{ij}}{d\theta}$$

Application: Optimal Taxation

- Efficiency component of a general: perturbation

$$\Xi^E = \sum_i \frac{\frac{dV^i}{d\theta}}{\lambda^i} = \sum_i \sum_j p^j \frac{dc^{ij}}{d\theta} + \sum_i \sum_j \tau^{ij} p^j \frac{dc^{ij}}{d\theta}$$

$$= \sum_j p^j \underbrace{\sum_i \frac{dc^{ij}}{d\theta}}_{=\frac{d\bar{y}^j}{d\theta}=0} + \sum_i \sum_j \tau^{ij} p^j \frac{dc^{ij}}{d\theta} = \sum_j p^j \sum_i \tau^{ij} \frac{dc^{ij}}{d\theta},$$

so

$$\Xi^E = 0 \quad \text{if} \quad \boxed{\tau^{ij} = \tau^j,\ \forall i} \rightarrow \boxed{\text{Uniform Consumption Tax}}$$

- As long as $\tau^{ij} = \tau^j$, $\forall i$, the economy will be efficient, since

$$\Xi^E = \sum_j p^j \tau^j \underbrace{\sum_i \frac{dc^{ij}}{d\theta}}_{=\frac{d\bar{y}^j}{d\theta}=0} = 0 \quad \text{whenever} \quad \tau^{ij} = \tau^j,\ \forall i$$

Application: Optimal Taxation

- Economic intuition: in a static exchange economy, distortions can only emerge if individuals exchange goods at different rates
  - If rates are different $\rightarrow$ cross-sectional exchange efficiency gains by reallocating consumption across individuals
- By ensuring that taxes for each good are identical across all individuals, economy is Pareto efficient
- Different welfarist planners may set different τ^j since these have different consequences for redistribution
  - But all of those allocations will be Pareto efficient is $\tau^{ij} = \tau^j, \forall i$
- **Remark:** this is a really easy problem
  - Insights become deeper with production

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Extra: Solving Consumer Problem via Perturbation

- **Problem**: $\max_{\{c^{ij}\}} u^i\left(\left\{c^{ij}\right\}_{j\in\mathcal{J}}\right)$ subject to $\sum_j p^j c^{ij} = w^i$
- Perturb objective $\rightarrow \frac{dV^i}{d\theta} = \sum_j \frac{\partial u^i}{\partial c^{ij}} \frac{dc^{ij}}{d\theta}$
- Perturb constraints $\rightarrow \sum_j p^j \frac{dc^{ij}}{d\theta} = 0$
- Take any two goods, say j and ℓ
  - Feasible perturbations must satisfy

$$p^j \frac{dc^{ij}}{d\theta} + p^\ell \frac{dc^{i\ell}}{d\theta} = 0 \Rightarrow \frac{dc^{i\ell}}{d\theta} = -\frac{p^j}{p^\ell}\frac{dc^{ij}}{d\theta}$$

- Change in objective is

$$\frac{dV^i}{d\theta} = \frac{\partial u^i}{\partial c^{ij}}\frac{dc^{ij}}{d\theta} + \frac{\partial u^i}{\partial c^{i\ell}}\frac{dc^{i\ell}}{d\theta} = \left(\frac{\partial u^i}{\partial c^{ij}} - \frac{\partial u^i}{\partial c^{i\ell}}\frac{p^j}{p^\ell}\right)\frac{dc^{ij}}{d\theta}$$

- Optimality conditions

To ensure that $\frac{dV^i}{d\theta} = 0$ for any feasible perturbation involving $\frac{dc^{ij}}{d\theta}$, it must be that

$$\frac{\partial u^i}{\partial c^{ij}} - \frac{\partial u^i}{\partial c^{i\ell}}\frac{p^j}{p^\ell} = 0 \Rightarrow \boxed{\frac{\frac{\partial u^i}{\partial c^{ij}}}{p^j} = \frac{\frac{\partial u^i}{\partial c^{i\ell}}}{p^\ell}} \leftarrow \text{Optimality Conditions}$$

Extra: Degrees of Freedom in Optimization

- Consumer Problem
  - J choices $-$ 1 constraint $= J - 1$ degrees of freedom
- Planning Problem
  - IJ choices $-$ J constraints $= IJ - J = (I-1)\,J$ degrees of freedom
  - Edgeworth Box: 2 degrees of freedom (efficiency and redistribution)

Extra: Open Ended Homework

1. Polemarchakis (1983)
   - Read the paper and rewrite it using the notation of the course
     - This is $I > 1$, $J = 2$ static exchange economy
   - Characterize and plot aggregate and individual excess demands in the (in the computer)
   - Characterize how a transfer (reallocation of initial endowments) changes allocations and prices (plot)
   - Illustrate the transfer paradox using the decomposition shown in class
   - Bonus: it it possible to get the same results with smooth (non-Leontief preferences)?
2. Transaction taxes
   - Consider transaction taxes, rather than consumption taxes

$$\sum_j p^j c^{ij} = \sum_j p^j \bar{y}^{ij} + T^i - \sum_j \tau^{ij} p^j \left| c^{ij} - \bar{y}^{ij} \right|$$

   - Present the efficiency implications of transaction taxes and characterize the taxes the minimize efficiency distortions
   - Compare the analysis with consumption taxes
   - It may be useful to look at Dávila (2023)