# ECON 500a
# General Equilibrium and Welfare Economics
# Competitive Equilibrium

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# Outline: Static Exchange Economies

1. Edgeworth Box Economy
2. Static Exchange Economy
3. Efficiency and Welfare
4. Microfounding Competition
5. $\boxed{\text{Competitive Equilibrium}}$

- Readings
  - MWG: 17.B through 17.I

# Outline: Competitive Equilibrium

1. $\boxed{\text{Existence and Computation}}$
2. Excess Demand Theorem
3. Uniqueness and Multiplicity
4. Convergence to Equilibria
5. Comparative Statics of Equilibrium Allocations and Prices
6. Economies with Many Individuals
7. Final Observations

# Reminder: Competitive/Walrasian Equilibrium

- A *competitive equilibrium* is an allocation

$$\mathring{\boldsymbol{c}} = \left\{ \underbrace{c^{11}, \ldots, c^{I1}}_{\text{good } 1}, \ldots, \underbrace{c^{1j}, \ldots, c^{Ij}}_{\text{good } j}, \ldots, \underbrace{c^{1J}, \ldots, c^{IJ}}_{\text{good } J} \right\}$$

and prices

$$\boldsymbol{p} = \left(p^1, \ldots, p^J\right)$$

such that

i) individuals choose consumption to maximize utility subject to their budget constraint taking prices as given:

$$\boldsymbol{c}^i\left(\boldsymbol{p}, \bar{\boldsymbol{y}}^i\right) = \arg\max_{\boldsymbol{c}^i} u^i\left(\left\{c^{ij}\right\}_{j \in \mathcal{J}}\right)$$

subject to

$$\sum_j p^j c^{ij} = \sum_j p^j \bar{y}^{ij}$$

ii) and markets clear, that is, resource constraints hold:

$$\sum_i c^{ij} = \sum_i \bar{y}^{ij}, \quad \forall j \in \mathcal{J}$$

# Existence Problem

- Does the model have a solution? Does an equilibrium $\boxed{\text{exist}}$?
  - For any set of primitives/parameters?
  - Under which conditions?
  - Starting point for a theory
- Major focus of early mathematical economics
  - Until proofs by Arrow and Debreu (1954) and McKenzie (1954)
  - See Düppe and Weintraub (2014) → Cowles Foundation
- I deemphasize existence relative to standard GE courses. Why?
  1. In practice, interested in model predictions for some parameters
  2. To find predictions, we must solve the model (analytically or in the computer) → ultimate guarantee than an equilibrium exists
  3. Many good treatments
     Debreu (1959), Arrow and Hahn (1971), Mas-Colell, Whinston, and Green (1995), Bewley (2007), Starr (2011), and Kreps (2013)

# Equation Counting

- Equilibrium prices solve nonlinear system:

$$\boldsymbol{z}\left(\boldsymbol{p}^{\star};\mathring{\bar{\boldsymbol{y}}}\right)=0$$

  - $J$ equations (aggregate excess demand for each good)
  - $J$ unknowns (prices for each good)
- In Edgeworth box economy, we can write
  dropping the dependence on endowments

$$z^{1}\left(p^{1\star},p^{2\star}\right)=0$$
$$z^{2}\left(p^{1\star},p^{2\star}\right)=0$$

- Two facts reduce the dimensionality of the system:
  i) Aggregate excess demand functions are homogeneous of degree zero allows us to normalize one price
     - Without loss of generality, we set good $J$ as numeraire: $p^{J}=1$
  ii) Walras' law allows us to conclude that as long as aggregate excess demand functions for $J-1$ goods are equal zero, the aggregate excess demand for the remaining good will also be zero

# Equation Counting

- We thus solve system of $J-1$ equations in $J-1$ unknowns:

$$\tilde{\boldsymbol{z}}\left(\boldsymbol{p}^{\star};\mathring{\tilde{\boldsymbol{y}}}\right)=0,$$

  where $\tilde{\boldsymbol{z}}\left(\boldsymbol{p}^{\star};\mathring{\tilde{\boldsymbol{y}}}\right)$ given by first $J-1$ elements of $\boldsymbol{z}\left(\boldsymbol{p}^{\star};\mathring{\tilde{\boldsymbol{y}}}\right)$
- In Edgeworth box economy:

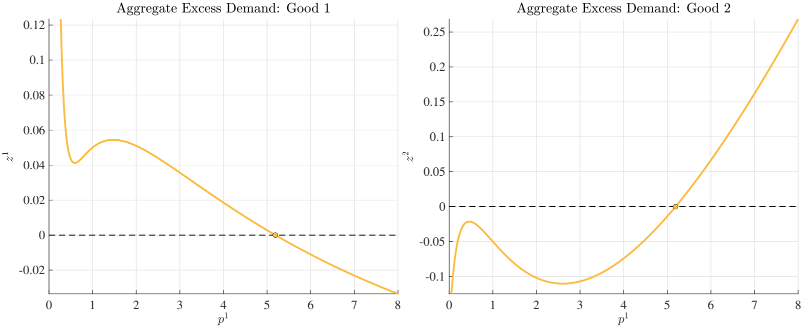
$$\tilde{\boldsymbol{z}}\left(\boldsymbol{p}^{\star};\mathring{\tilde{\boldsymbol{y}}}\right)=z^{1}\left(p^{1\star},1;\mathring{\tilde{\boldsymbol{y}}}\right)=0$$

- If we had a linear system $\rightarrow$ easy to check whether one, many, or no solutions exist
- *Nonlinear* equations may have no (real) solutions
  - e.g. $x^2+1=0$ or $x^2+x+1=0$
- $\boxed{\text{Equation counting:}}$ if number of equations is equal to the number of unknowns $\rightarrow$ we will typically find a solution
  - Very useful in practice! (not in theory)
  - Generically works

# Existence with $J = 2$ via Excess Demand Diagram

- Excess demand diagram works with $I > 1$ as long as $J = 2$
  - $p^2 = 1$ and plot $p^1$
- $J = 2 \rightarrow$ existence proof follows from intermediate value theorem
  Bolzano's theorem
  - If a continuous function has values of opposite sign in an interval, then it must have a root in that interval
- Aggregate excess demands are continuous
- Enough to show that $z^1\left(p^{1\star}\right)$ takes positive and negative values
  1. As $p^1 \rightarrow 0$, strong monotonicity implies that $z^1\left(p^{1\star}\right) \rightarrow \infty$
  2. As $p^1 \rightarrow \infty$ (equivalently, $p^2 \rightarrow 0$), the same argument and Walras' law ensures that $z^1\left(p^{1\star}\right)$ converges to a negative number
  3. Since $c^{ij} \geq 0$, excess demands are bounded below by the aggregate endowments of goods
- This proof illustrates economics behind many existence proofs
  1. Establish continuity
  2. Take care of behavior at boundaries

# Existence with $J = 2$ via Excess Demand Diagram



# Fixed-Point Proofs vs. Computation

- Classical existence proofs are based on fixed point theorems
  Arrow and Debreu (1954) and McKenzie (1954)
- Define continuous and convex mapping between an non-empty, compact, and convex set (of prices) into itself
  - Use fixed point theorem (Brouwer or Kakutani's)
- Drawback of proofs based on fixed-point theorems
  - They are not constructive
  - How do I compute the equilibrium?
- Question of $\boxed{\text{computation}}$ $\rightarrow$ Scarf (1973)

# Non-Existence

- What if a competitive equilibrium does not exist?
  - No allocations and prices that satisfy our definition
  - Inconsistent economic structure
- Two threats to the existence of equilibrium
  1. Left: Lack of Strong Monotonicity
  2. Right: Non-Convexity

# Computation: Negishi Approach

- Planning problems easier to compute than competitive equilibria
  - $IJ$ variables vs $IJ + J$
- $\boxed{\text{Negishi (1960)}}$: use planning problem to characterize CE
  - Based on second welfare theorem
- Negishi approach $\rightarrow$ Computational advantage
  - Very common in neoclassical macro $\Rightarrow$ RBC Model
- Drawback of this approach
  - Does not work with frictions (e.g. taxes, market power, etc.)
  - Excess demand approach always works

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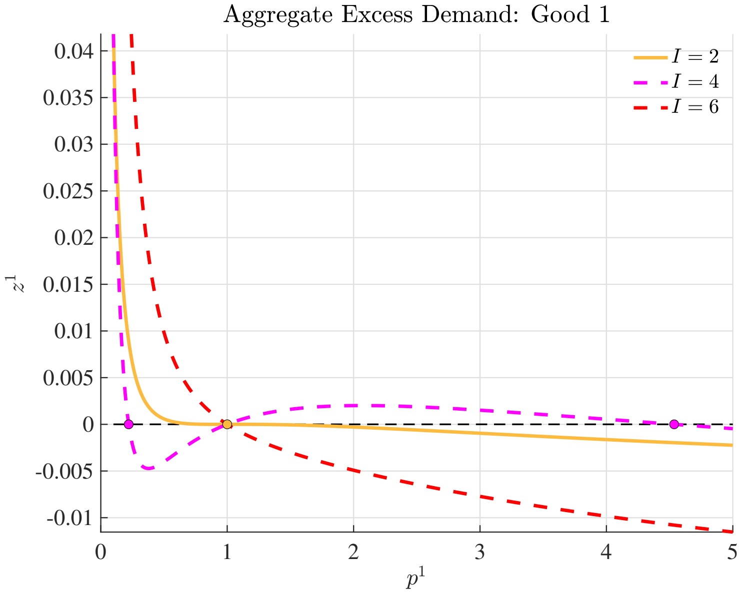
# Excess Demand Theorem: Sonnenschein-Mantel-Debreu

- Individual excess demands restricted by individual optimization
- Excess Demand Theorem: aggregate excess demand functions that satisfy continuity, homogeneity of degree zero, and Walras's law can take almost any shape
- Remarks:
  - Large $I$ may be need (more degrees of freedom)
  - This occurs because of income/wealth effects
- Result often interpreted as "competitive model has no testable predictions"
  - This is not right
  - Theory has huge explanatory power! (even without information, imperfect competition, etc.)
- We simply have to put more discipline
  1. Which endowments do agents have?
  2. Which preferences?
- See Brown and Matzkin (1996) and Kübler and Polemarchakis (2024) for testability/recoverability in GE

# Excess Demand Theorem: Sonnenschein-Mantel-Debreu



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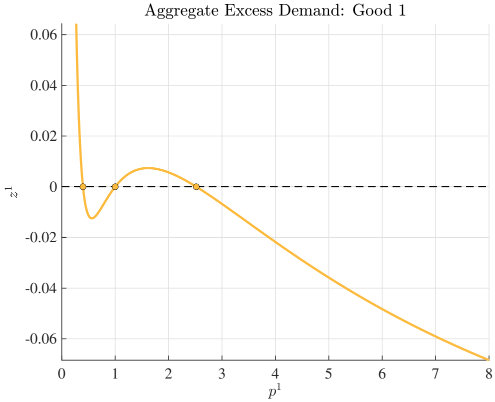
# Uniqueness and Multiplicity

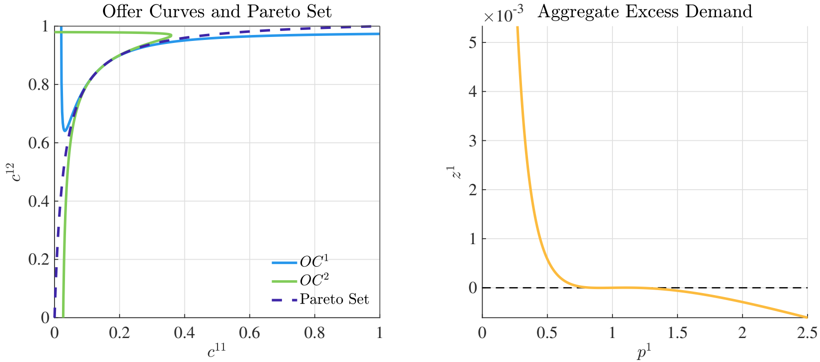
- Good theories have unique predictions
  - Walrasian economies typically have multiple equilibria
  - Corollary of the excess demand theorem
- In practice: Cobb Douglas or CES → unique equilibrium
  - *Gross substitutes* property: $\partial z^j / \partial p^\ell > 0$ for $j \neq \ell$
  - Intuition: demands must slope down sufficiently




- Toda and Walsh (2024) → recent survey of uniqueness

# Local Uniqueness

- How about *local uniqueness*?
  - An equilibrium is locally unique (or isolated) if we cannot find another normalized equilibrium price vector arbitrarily close to it
- This is useful so theory is locally determinate
  - Maybe economy ended up around one equilibrium




- This case looks pathological

# Regular Economies and Multiplicity

- Remember: $\tilde{\boldsymbol{z}}\left(\boldsymbol{p}^{\star}; \mathring{\bar{\boldsymbol{y}}}\right) = 0$
- An *equilibrium* is *regular* if Jacobian $\frac{\partial \tilde{\boldsymbol{z}}}{\partial \boldsymbol{p}}$ has full rank $(J-1)$
  - If every equilibrium is regular, the *economy* is *regular*
- The following properties hold for regular economies
  1. Regular (normalized) equilibrium price vectors are *locally unique/isolated*
  2. Regular economies have a *finite* number of (normalized) equilibria
  3. Regular economies have an *odd* number of (normalized) equilibria
  4. Equilibria in which aggregate excess demand "slopes down" have to exist

# Index Theorem

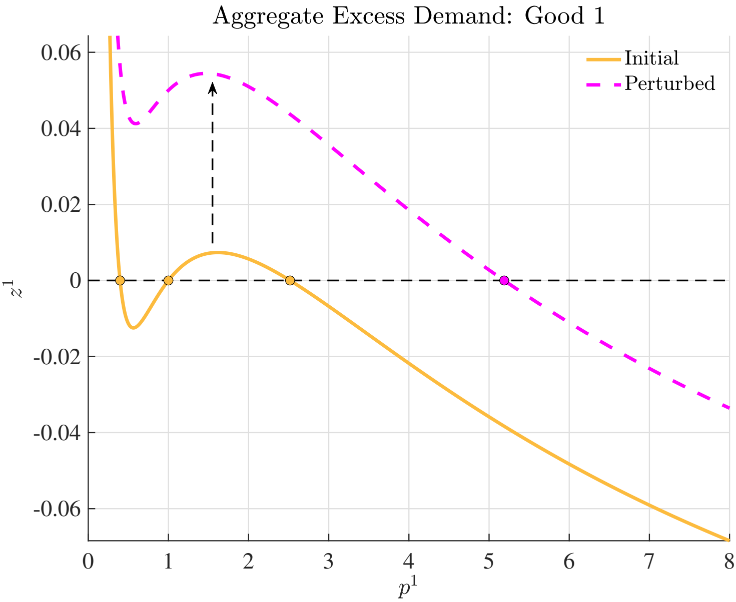
- For a regular economy

$$\sum_{\{\boldsymbol{p}: z(\boldsymbol{p})=0,\ p^J=1\}} \text{index } p = +1,$$

where index $p = (-1)^{J-1} \operatorname{sgn} \left| \frac{\partial \tilde{\boldsymbol{z}}}{\partial \boldsymbol{p}} \right|$, where $\left| \frac{\partial \tilde{\boldsymbol{z}}}{\partial \boldsymbol{p}} \right|$ is the determinant of the $(J-1) \times (J-1)$ matrix $\frac{\partial \tilde{\boldsymbol{z}}}{\partial \boldsymbol{p}}$.

- In $J = 2$ case $\rightarrow$ index $p$ is
  - positive for equilibrium with downward-sloping demand
  - negative for equilibrium with upward-sloping demand
- All economies are generically regular
  - Transversality (Sard's Theorem)
  - See Mas-Colell (1985)

# Economies are Generically Regular



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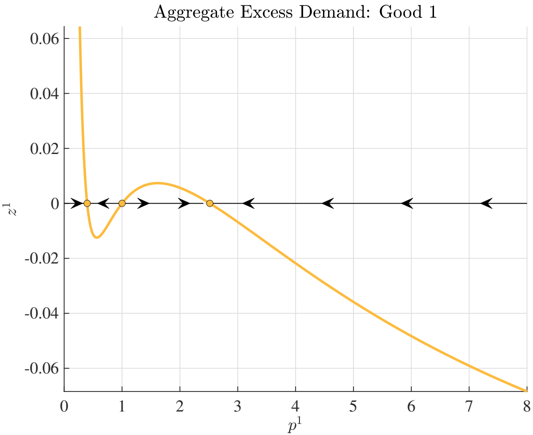
# Convergence to Equilibria

- How is equilibrium achieved? What if the economy is in disequilibrium?
- Price tâtonnement:
  - Postulate the prices evolve by
    $\zeta^j > 0$ are parameters

$$\frac{dp^j}{dt} = \zeta^j z^j \left(\boldsymbol{p}\left(t\right); \bar{\boldsymbol{y}}\right)$$

  - Time: $t \in [0, \infty]$
- System of ODE's: initial price vector $\boldsymbol{p}\left(0\right)$
- Tâtonnement is reasonable:
  - If $z^j > 0$ (excess demand) $\Rightarrow$ increase the price
  - If $z^j < 0$ (excess supply) $\Rightarrow$ lower the price
- Problems:
  - Who changes the price? "Walrasian auctioneer"
  - Why all individuals trade at the same price out of equilibrium?

# $J = 2$ Case




- An <u>equilibrium is stable</u> if the economy converges to it
  given initial price vector and tâtonnement rule
  - *Globally* stable: for any initial point
  - *Locally* stable: for sufficiently close points
- An <u>economy is stable</u>, if for any initial point, the economy converges to an equilibrium

# Tâtonnement

- For $J = 2$, the economy is stable
  - Easy to show given what we know
- For $J > 2$, anything can happen
  - Famous $J = 3$ example by Scarf (1960) $\rightarrow$ coming next
  - Features cycles $\rightarrow$ economy is unstable
- Tâtonnement literature died
- Somewhat related work on
  - Learnability (cobweb dynamics)
  - Disequilibrium

# Scarf 1960 Example

- $I = J = 3$
- Preferences and endowments given by
  1. $V^1 = \min\left\{c^{11}, c^{12}\right\}$ and $\bar{\boldsymbol{y}}^1 = (1, 0, 0)$
  2. $V^2 = \min\left\{c^{22}, c^{23}\right\}$ and $\bar{\boldsymbol{y}}^2 = (0, 1, 0)$
  3. $V^3 = \min\left\{c^{31}, c^{33}\right\}$ and $\bar{\boldsymbol{y}}^3 = (0, 0, 1)$
- Demand and excess demand for individual 1 given by

$$\boldsymbol{c}^1\left(\boldsymbol{p}, \bar{\boldsymbol{y}}^i\right) = \begin{pmatrix} \frac{p^1 \bar{y}^1}{p^1 + p^2} \\ \frac{p^1 \bar{y}^1}{p^1 + p^2} \\ 0 \end{pmatrix} \Rightarrow \boldsymbol{z}^1\left(\boldsymbol{p}, \bar{\boldsymbol{y}}^i\right) = \begin{pmatrix} \frac{p_1}{p_1 + p_2} - 1 \\ \frac{p_1}{p_1 + p_2} \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{-p_2}{p_1 + p_2} \\ \frac{p_1}{p_1 + p_2} \\ 0 \end{pmatrix}$$

  - Similar for $i = 2$ and $i = 3 \rightarrow$ Derive it!
- Aggregate excess demand:

$$\boldsymbol{z}\left(\boldsymbol{p}; \mathring{\bar{\boldsymbol{y}}}\right) = \begin{pmatrix} \frac{-p_2}{p_1 + p_2} + \frac{p_2}{p_3 + p_1} \\ \frac{-p_3}{p_2 + p_3} + \frac{p_1}{p_1 + p_2} \\ \frac{-p_1}{p_3 + p_1} + \frac{p_2}{p_3 + p_3} \end{pmatrix}$$

# Scarf 1960 Example

- Tâtonnement adjustment:

$$\frac{dp^j}{dt} = z^j\left(\boldsymbol{p}\left(t\right);\bar{\boldsymbol{y}}\right),\ \forall j$$

- Walras' law implies that $\sum_j \left(p^j\right)^2$ is constant, since its time derivative is zero:

$$\sum_j p^j\left(t\right) z^j\left(\boldsymbol{p}\left(t\right);\bar{\boldsymbol{y}}\right) = \sum_j p^j \frac{dp^j}{dt} = \frac{d\left(\sum_j \left(p^j\right)^2\right)}{dt} = 0$$

- Unique equilibrium: $p^1 = p^2 = p^3 \rightarrow$ solves $\boldsymbol{z}\left(\boldsymbol{p};\mathring{\bar{\boldsymbol{y}}}\right) = 0$
- Only solution to tâtonnement ODE is (see Scarf (1960))

$$p^1p^2p^3 = \text{constant}$$

  - That is, prices will not adjust
- Without loss of generality, choose $\sum_j \left(p^j\right)^2 = 3$
- Unless we start from $p^1p^2p^3 = 1 \rightarrow$ economy is unstable

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# Comparative Statics

- We can characterize $\frac{dc^{ij}}{d\theta}$ via comparative statics
  - Useful for testable predictions
  - Useful for welfare assessments
- Formally, using implicit function theorem:
  $\frac{\partial \tilde{\boldsymbol{z}}}{\partial \boldsymbol{p}}$ and $\frac{\partial \tilde{\boldsymbol{z}}}{\partial \mathring{\bar{\boldsymbol{y}}}}$ are Jacobian matrices

$$\tilde{\boldsymbol{z}}\left(\boldsymbol{p}^\star; \mathring{\bar{\boldsymbol{y}}}\right) = 0 \Rightarrow \frac{\partial \tilde{\boldsymbol{z}}}{\partial \boldsymbol{p}} \frac{d\boldsymbol{p}^\star}{d\theta} + \frac{\partial \tilde{\boldsymbol{z}}}{\partial \mathring{\bar{\boldsymbol{y}}}} \frac{d\mathring{\bar{\boldsymbol{y}}}^\star}{d\theta} = 0 \Rightarrow \boxed{\frac{d\boldsymbol{p}^\star}{d\theta} = \left(-\frac{\partial \tilde{\boldsymbol{z}}}{\partial \boldsymbol{p}}\right)^{-1} \frac{\partial \tilde{\boldsymbol{z}}}{\partial \mathring{\bar{\boldsymbol{y}}}} \frac{d\mathring{\bar{\boldsymbol{y}}}^\star}{d\theta}}$$

- We need $\frac{\partial \tilde{\boldsymbol{z}}}{\partial \boldsymbol{p}}$ to have full rank $\rightarrow$ Regular equilibrium!
- Given $\frac{d\boldsymbol{p}^\star}{d\theta}$:

$$\frac{d\boldsymbol{c}^i\left(\boldsymbol{p}^\star, \bar{\boldsymbol{y}}^i\right)}{d\theta} = \frac{\partial \boldsymbol{c}^i}{\partial \boldsymbol{p}^\star} \frac{d\boldsymbol{p}^\star}{d\theta} + \frac{\partial \boldsymbol{c}^i}{\partial \bar{\boldsymbol{y}}^i} \frac{d\bar{\boldsymbol{y}}^i}{d\theta},$$

- Hard to obtain general analytical comparative statics
  - Functional form assumptions?
  - Sufficient statistics?
- We can also parametrize preferences or other exogenous elements of more general models, for instance, taxes, technologies, etc.

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# Economies with Many Individuals

- Individual non-convexities can still yield convexity in aggregate as $I \to \infty$
- *Shapley-Folkman-Starr* theorem
  - Minkowski sum (or average) of a large number of non-convex sets is approximately convex
- It is common to assume that we have a *continuum* of agents
  - Existence ensured by continuum: Aumann (1966)

# Illustrating the Continuum

- Economy with discontinuous (bang-bang) demand:

$$c^{i1} = \begin{cases} 0, & \text{if } p^1 > \bar{p}^{1i} \\ 1, & \text{if } p^1 \leq \bar{p}^{1i} \end{cases}$$

  - $\bar{p}^{1i}$ is an individual specific threshold
- Distribution of thresholds $\bar{p}^i$ has a continuous cdf $F(\cdot)$
- Aggregate demand is

$$c^1\left(p^1\right) = \int c^{i1} di = \int_0^{p^1} dF\left(\bar{p}^i\right) = F\left(p^1\right)$$

  - $c^1\left(p^1\right)$ is continuous but $c^{i1}$ is not!
- Continuum models are popular
  - Dornbusch, Fischer, and Samuelson (1977)
  - Geanakoplos (2010)

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# Final Observations I

1. Characterizing competitive equilibria amounts to solving a system of nonlinear equations. Under reasonably assumptions (mostly continuity and convexity), this system will have a solution and competitive equilibria will exist.
2. Although most models used in practice feature a unique equilibrium, we should not be surprised to find models with multiple equilibria. Multiple equilibria are due to strong income effects.
3. Whenever a competitive economy has multiple equilibria, these cannot be Pareto ranked.
4. Generically, the number of equilibria is finite and odd. Equilibria are locally isolated.
5. By aggregating enough individual excess demands we can find aggregate excess demands with any form, so the set of equilibrium prices can in general take any form. Once again, these phenomena are due to strong income effects.

# Final Observations II

6. Similarly, the out-of-equilibrium convergence properties of the model can take any form. It is possible to have a unique equilibrium to which it is not possible to converge under a reasonable tâtonnement (adjustment) procedure.
7. Obviously, general equilibrium models are testable. Testing models may require to conduct comparative static exercises, which are based on derivatives of the aggregate excess demand function.
8. As the number of individuals in an economy gets large, non-convexities vanish in the aggregate, ensuring for instance existence.

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