ECON 500a
General Equilibrium and Welfare Economics
Efficiency and Welfare: Production Economies

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Outline: Static Production Economies

1. Elementary Static Production Economies
2. General Static Production Economy
3. Efficiency and Welfare
4. Applications

Outline: Efficiency and Welfare

1. Welfare Assessments
2. Planning Problem
3. Welfare Theorems

Efficiency/Redistribution Decomposition

- Given a physical structure, can we systematically attribute the welfare gains of a perturbation to specific sources?
  - Question of "*Origins of welfare gains*"
  - Now with production
- Recall from static exchange:

$$\frac{dW^{\lambda}}{d\theta} = \frac{\frac{dW}{d\theta}}{\frac{1}{I}\sum_i \frac{\partial \mathcal{W}}{\partial V^i}\lambda^i} = \sum_i \omega^i \frac{dV^{i|\lambda}}{d\theta} \quad \text{where} \quad \omega^i = \frac{\frac{\partial \mathcal{W}}{\partial V^i}\lambda^i}{\frac{1}{I}\sum_i \frac{\partial \mathcal{W}}{\partial V^i}\lambda^i}$$

  - λ^i normalizing factor to choose numeraire

- Efficiency-Redistribution decomposition:

$$\frac{dW^{\lambda}}{d\theta} = \underbrace{\sum_i \frac{dV^{i|\lambda}}{d\theta}}_{\Xi^E \text{ (Efficiency)}} + \underbrace{\mathbb{C}ov_i^{\Sigma}\left[\omega^i, \frac{dV^{i|\lambda}}{d\theta}\right]}_{\Xi^{RD} \text{ (Redistribution)}}$$

Decomposing Efficiency

- Preferences $V^i = u^i\left(\left\{c^{ij}\right\}_{j\in\mathcal{J}}, \left\{n^{if,s}\right\}_{f\in\mathcal{F}}\right)$ imply that

$$\begin{aligned}\frac{\frac{dV^i}{d\theta}}{\lambda^i} &= \sum_j \frac{\frac{\partial u^i}{\partial c^{ij}}}{\lambda^i}\frac{dc^{ij}}{d\theta} + \sum_f \frac{\frac{\partial u^i}{\partial n^{if,s}}}{\lambda^i}\frac{dn^{if,s}}{d\theta} \\ &= \sum_j MRS_c^{ij}\frac{dc^{ij}}{d\theta} - \sum_f MRS_n^{if}\frac{dn^{if,s}}{d\theta}\end{aligned}$$

  where

$$MRS_c^{ij} = \frac{\frac{\partial u^i}{\partial c^{ij}}}{\lambda^i} \quad \text{and} \quad MRS_n^{if} = -\frac{\frac{\partial u^i}{\partial n^{if,s}}}{\lambda^i}$$

- Individual welfare gains due to changes in consumption and factor supply

Decomposing Efficiency

- Efficiency:

$$\Xi^E = \sum_i \frac{\frac{dV^i}{d\theta}}{\lambda^i} = \sum_j \sum_i MRS_c^{ij} \frac{dc^{ij}}{d\theta} - \sum_f \sum_i MRS_n^{if} \frac{dn^{if,s}}{d\theta}$$

- Define shares $c^{ij} = \chi_c^{ij} c^j$ and $n^{if,s} = \chi_n^{if,s} n^{f,s}$, so

$$\frac{dc^{ij}}{d\theta} = \frac{d\chi_c^{ij}}{d\theta} c^j + \chi_c^{ij} \frac{dc^j}{d\theta} \quad \text{and} \quad \frac{dn^{if,s}}{d\theta} = \frac{d\chi_n^{if,s}}{d\theta} n^{f,s} + \chi_n^{if,s} \frac{dn^{f,s}}{d\theta}$$

- Therefore

$$\sum_i MRS_c^{ij} \frac{dc^{ij}}{d\theta} = \mathbb{C}ov_i^{\Sigma} \left[MRS_c^{ij}, \frac{d\chi_c^{ij}}{d\theta} \right] c^j + AMRS_c^j \frac{dc^j}{d\theta}$$

$$\sum_i MRS_n^{if,s} \frac{dn^{if,s}}{d\theta} = \mathbb{C}ov_i^{\Sigma} \left[MRS_n^{if,s}, \frac{d\chi_n^{if,s}}{d\theta} \right] n^{f,s} + AMRS_n^f \frac{dn^{f,s}}{d\theta}$$

- Define aggregate marginal rates of substitution (AMRS):

$$AMRS_c^j = \sum_i \chi_c^{ij} MRS_c^{ij} \quad \text{and} \quad AMRS_n^f = \sum_i \chi_n^{if,s} MRS_n^{if,s}$$

Welfare Assessments: Exchange + Production Efficiency

- $\Xi^{E} = \Xi^{AE,X} + \Xi^{AE,P}$: Efficiency $\to$ Exchange + Production
- Exchange efficiency:

$$\Xi^{AE,X} = \underbrace{\mathbb{C}ov_i^{\Sigma}\left[MRS_c^{ij}, \frac{d\chi_c^{ij}}{d\theta}\right] c^j}_{\substack{\text{Cross-Sectional} \\ \text{Consumption Efficiency}}} \underbrace{- \mathbb{C}ov_i^{\Sigma}\left[MRS_n^{if,s}, \frac{d\chi_n^{if,s}}{d\theta}\right] n^{f,s}}_{\substack{\text{Cross-Sectional} \\ \text{Factor Supply Efficiency}}}$$

  - Welfare gains from reallocating consumption and factor supply
  - If $I = 1 \Rightarrow \Xi^{AE,X} = 0$, but different from redistribution!
- Production efficiency:

$$\Xi^{AE,P} = \sum_j AMRS_c^j \frac{dc^j}{d\theta} - \sum_f AMRS_n^f \frac{dn^{f,s}}{d\theta}$$

  - Welfare gains from consuming more (net of cost of supplying factors)

Production Efficiency

- Production function: $y^{j,s} = G^j\left(\left\{n^{jf,d}\right\}_{f\in\mathcal{F}}; \theta\right)$
  - No intermediate uses

    Intermediates → Dávila and Schaab (2024)

$$\frac{dy^{j,s}}{d\theta} = \sum_f \frac{\partial G^j}{\partial n^{jf,d}} \frac{dn^{jf,d}}{d\theta} + \frac{\partial G^j}{\partial \theta}$$

- Factor use share: $n^{jf,d} = \chi_n^{jf,d} n^{f,d}$

$$\frac{dn^{jf,d}}{d\theta} = \frac{d\chi_n^{jf}}{d\theta} n^{f,d} + \chi_n^{jf,d} \frac{dn^{f,d}}{d\theta}$$

- Consumption perturbation:

$$\begin{aligned}
\frac{dc^j}{d\theta} &= \frac{dy^{j,s}}{d\theta} + \frac{d\bar{y}^{j,s}}{d\theta} \\
&= \sum_f \frac{\partial G^j}{\partial n^{jf,d}} \left(\frac{d\chi_n^{jf}}{d\theta} n^{f,d} + \chi_n^{jf,d} \frac{dn^{f,d}}{d\theta}\right) + \frac{\partial G^j}{\partial \theta} + \frac{d\bar{y}^{j,s}}{d\theta} \\
&= \sum_f \frac{\partial G^j}{\partial n^{jf,d}} \frac{d\chi_n^{jf}}{d\theta} n^{f,d} + \sum_f \chi_n^{jf,d} \frac{\partial G^j}{\partial n^{jf,d}} \frac{dn^{f,d}}{d\theta} + \frac{\partial G^j}{\partial \theta} + \frac{d\bar{y}^{j,s}}{d\theta}
\end{aligned}$$

Production Efficiency

$$\sum_j AMRS_c^j \frac{dc^j}{d\theta} = \sum_f \left(\sum_j \underbrace{AMRS_c^j \frac{\partial G^j}{\partial n^{jf,d}}}_{=MWP_n^{jf}} \frac{d\chi_n^{jf}}{d\theta} \right) n^{f,d}$$

$$+ \sum_f \left(\underbrace{\sum_j \chi_n^{jf,d} AMRS_c^j \frac{\partial G^j}{\partial n^{jf,d}}}_{=AMWP_n^f} \right) \frac{dn^{f,s}}{d\theta}$$

$$+ \sum_j AMRS_c^j \left(\frac{\partial G^j}{\partial \theta} + \frac{d\bar{y}^{j,s}}{d\theta} \right) + \sum_f AMWP_n^f \frac{d\bar{n}^{f,s}}{d\theta}$$

- MWP_n^{jf}: welfare gain from adjusting factor
- $AMWP_n^f$: welfare gain from using extra unit of factor (for shares $\chi_n^{jf,d}$)

Production Efficiency

- From $\Xi^{AE,P} = \sum_j AMRS_c^j \frac{dc^j}{d\theta} - \sum_f AMRS_n^f \frac{dn^{f,s}}{d\theta}$ to

$$\Xi^{AE,P} = \underbrace{\overbrace{\sum_f \mathbb{C}ov_j^{\Sigma}\left[MWP_n^{jf}, \frac{d\chi_n^{jf}}{d\theta}\right] n^{f,d}}^{\substack{\text{Cross-Sectional} \\ \text{Factor Efficiency}}} + \overbrace{\sum_f \left(AMWP_n^f - AMRS_n^f\right)\frac{dn^{f,s}}{d\theta}}^{\substack{\text{Aggregate} \\ \text{Factor Efficiency}}}}_{\text{Factor Efficiency}}$$

$$+ \underbrace{\sum_j AMRS_c^j \frac{\partial G^j}{\partial \theta}}_{\substack{\text{Technology} \\ \text{Change}}} + \underbrace{\sum_j AMRS_c^j \frac{d\bar{y}^{j,s}}{d\theta}}_{\substack{\text{Good Endowment} \\ \text{Change}}} + \underbrace{\sum_f AMWP_n^f \frac{d\bar{n}^{f,s}}{d\theta}}_{\substack{\text{Factor Endowment} \\ \text{Change}}}$$

- XSFE: Welfare gains from reallocating factors across uses
  - Horizontal economy
- AFE: Welfare gains from adjusting aggregate factor supply
  - Robinson Crusoe economy
- Other three terms: changes in endowments or technology
- **No assumptions on economic structure!**

Outline: Efficiency and Welfare

1. Welfare Assessments
2. Planning Problem
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Planning Problem: Perturbation

- Perturbation approach $\rightarrow$ easiest

1. Exchange efficiency:
    - Consumption $\rightarrow MRS_c^{ij}$ equalized across individuals
        - $MRS_c^{ij} = \frac{\frac{\partial u^i}{\partial c^{ij}}}{\lambda^i} = \frac{\frac{\partial u^m}{\partial c^{mj}}}{\lambda^m} = MRS_c^{mj}$
        - This implies that $AMRS_c^j = \sum_i \chi_c^{ij} MRS_c^{ij} = MRS_c^{ij}$, for any i
    - Factor supply $\rightarrow MRS_n^{if,s}$ equalized across individuals
        - $MRS_n^{if} = -\frac{\frac{\partial u^i}{\partial n^{if,s}}}{\lambda^i} = -\frac{\frac{\partial u^m}{\partial n^{mf,s}}}{\lambda^m} = MRS_n^{mf}$
        - This implies that $AMRS_n^f = \sum_i \chi_n^{if,s} MRS_n^{if,s} = MRS_n^{if,s}$, for any i

Planning Problem: Perturbation

2. Production efficiency:
   - Cross-sectional factor $\rightarrow MWP_n^{jf}$ equalized across uses
     - $MWP_n^{jf} = AMRS_c^j \frac{\partial G^j}{\partial n^{jf,d}} = AMRS_c^\ell \frac{\partial G^\ell}{\partial n^{\ell f,d}} = MWP_n^{\ell f}$
     - This implies that $AMWP_n^f = \sum_j \chi_n^{jf,d} MWP_n^{jf} = MWP_n^{jf}$
   - Aggregate factor $\rightarrow AMWP_n^f = AMRS_n^f$, which in turn ensures that

$$MWP_n^{jf} = MRS_n^{if,s},$$

for any j, i, f combination.

Planning Problem: Lagrangian

$$\begin{aligned}
\mathcal{L} = & \sum_i \alpha^i u^i \left(\left\{ c^{ij} \right\}_{j \in \mathcal{J}}, \left\{ n^{if,s} \right\}_{f \in \mathcal{F}} \right) \\
& - \sum_j \eta_y^j \left(\sum_i c^{ij} - G^j \left(\left\{ n^{jf,d} \right\}_f \right) \right) - \sum_f \eta_n^f \left(\sum_j n^{jf,d} - \sum_i n^{if,s} - \sum_i \bar{n}^{if,s} \right) \\
& + \sum_i \sum_j \kappa_c^{ij} c^{ij} + \sum_i \sum_f \kappa_n^{if,s} n^{if,s} + \sum_j \sum_f \kappa_n^{jf,d} n^{jf,d},
\end{aligned}$$

- Optimality: same arguments as exchange

$$\begin{aligned}
\frac{d\mathcal{L}}{dc^{ij}} &= \alpha^i \frac{\partial u^i}{\partial c^{ij}} - \eta_y^j + \kappa_c^{ij} = 0 \\
\frac{d\mathcal{L}}{dn^{if,s}} &= \alpha^i \frac{\partial u^i}{\partial n^{if,s}} + \eta_n^f + \kappa_n^{if,s} = 0 \\
\frac{d\mathcal{L}}{dn^{jf,d}} &= \eta_y^j \frac{\partial G^j}{\partial n^{jf,d}} - \eta_n^f + \kappa_n^{jf,d} = 0.
\end{aligned}$$

Outline: Efficiency and Welfare

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Proof #1 of First Welfare Theorem I

- Consider CE → suppose another feasible allocation Pareto dominates it
- The strictly better off individual could not have afforded the new allocation at competitive prices, so

$$\sum_j p^{j\star} c^{ij} > \sum_j p^{j\star} \bar{y}^{ij,s} + \sum_f w^{f\star} \left(n^{if,s\star} + \bar{n}^{if,s} \right) + \sum_j \nu^{ij} \pi^{j\star}$$

- Local non-satiation ensures that, for all other individuals:

$$\sum_j p^{j\star} c^{ij} > \sum_j p^{j\star} \bar{y}^{ij,s} + \sum_f w^{f\star} \left(n^{if,s\star} + \bar{n}^{if,s} \right) + \sum_j \nu^{ij} \pi^{j\star}$$

- Aggregating

$$\sum_i \sum_j p^{j\star} c^{ij} > \sum_i \sum_j p^{j\star} \bar{y}^{ij,s} + \underbrace{\sum_i \sum_f w^{f\star} \left(n^{if,s\star} + \bar{n}^{if,s} \right) + \sum_j \overbrace{\sum_i \nu^{ij}}^{=1} \pi^{j\star}}_{=\sum_j p^{j\star} y^{j,s}}$$

Proof #1 of First Welfare Theorem II

- So
$$\sum_j p^{j\star}\left(c^j - \bar{y}^{j,s} - y^{j,s}\right) > 0$$
- But market clearing requires $c^j = \sum_i c^{ij} = \sum_i \bar{y}^{ij,s} + y^{j,s} = \bar{y}^{j,s} + y^{j,s}$, which contradicts the previous equation
  - Hence, no feasible allocation can Pareto dominate a CE
  - Any competitive equilibrium is Pareto efficient

Proof #2 of Second Welfare Theorem I

- Consider interior case (can be relaxed)
- Individual optimality conditions

  λ^i: Lagrange multiplier on budget constraint

$$\frac{\partial u^i}{\partial c^{ij}} - \lambda^i p^j = 0 \quad \text{and} \quad -\frac{\partial u^i}{\partial n^{if,s}} - \lambda^i w^f = 0$$

- Planning optimality conditions

  α^i: Pareto weight, η_y^j: good j's Lagrange multiplier, and η_n^f: factor f's Lagrange multiplier

$$\frac{\partial u^i}{\partial c^{ij}} - \frac{1}{\alpha^i}\eta_y^j = 0 \quad \text{and} \quad -\frac{\partial u^i}{\partial n^{if,s}} - \frac{1}{\alpha^i}\eta_n^f = 0$$

- Production side: competition

$$p^j \frac{\partial G^j}{\partial x^{j\ell}} - w^f = 0$$

Proof #2 of Second Welfare Theorem II

- Production side: planning

$$\eta_y^j \frac{\partial G^j}{\partial n^{j\,f,d}} - \eta_n^f = 0$$

- One-to-one mappings between λ^i and α^i, between η_y^j and p^j, and η_n^f and w^f:

$$\lambda^i \leftrightarrow \frac{1}{\alpha^i}, \qquad p^j \leftrightarrow \eta_y^j, \quad \text{and} \quad w^f \leftrightarrow \eta_n^f$$

- Given a CE, if we choose Pareto weights $\alpha^i = \frac{1}{\lambda^i}$, we know that $p^j = \eta_n^j$ and $w^f = \eta_n^f$is a solution of the planning problem
  - Any competitive equilibrium is Pareto efficient
- We get second welfare theorem for free!

Proof #3 of First Welfare Theorem I

- Starting from a CE, compute individual welfare gains of a perturbation:

  λ^i: Lagrange multiplier on budget constraint

$$
\begin{aligned}
\frac{dV^i}{d\theta} &= \lambda^i \left(\sum_j \frac{\frac{\partial u^i}{\partial c^{ij}}}{\lambda^i} \frac{dc^{ij}}{d\theta} + \sum_f \frac{\frac{\partial u^i}{\partial n^{if,s}}}{\lambda^i} \frac{dn^{if,s}}{d\theta} \right) \\
&= \lambda^i \left(\sum_j p^j \frac{dc^{ij}}{d\theta} - \sum_f w^f \frac{dn^{if,s}}{d\theta} \right)
\end{aligned}
$$

  - Last equation uses individual optimality

- Say we perturb individual demands, but budget constraints and market clearing remain satisfied:

$$
\begin{aligned}
\sum_j p^j \frac{dc^{ij}}{d\theta} - \sum_f w^f \frac{dn^{if,s}}{d\theta} = \sum_j \frac{dp^j}{d\theta} \left(\bar{y}^{ij,s} - c^{ij} \right) \\
+ \sum_f \frac{dw^f}{d\theta} \left(n^{if,s} + \bar{n}^{if,s} \right) + \sum_j \nu^{ij} \frac{d\pi^j}{d\theta}
\end{aligned}
$$

Proof #3 of First Welfare Theorem II

- Profits must also adjust:

$$
\begin{aligned}
\frac{d\pi^j}{d\theta} &= p^j \frac{dy^{j,s}}{d\theta} - \sum_f w^f \frac{dn^{jf,d}}{d\theta} + \frac{dp^j}{d\theta} y^{j,s} - \sum_f \frac{dw^f}{d\theta} n^{jf,d} \\
&= \sum_f \underbrace{\left(p^j \frac{\partial G^j}{\partial n^{jf,d}} - w^f \right)}_{=0} \frac{dn^{jf,d}}{d\theta} + \frac{dp^j}{d\theta} y^{j,s} - \sum_f \frac{dw^f}{d\theta} n^{jf,d}
\end{aligned}
$$

- We can express the normalized individual welfare gain $\frac{\frac{dV^i}{d\theta}}{\lambda^i}$ as

$$
\begin{aligned}
\frac{\frac{dV^i}{d\theta}}{\lambda^i} = \sum_j p^j \frac{dc^{ij}}{d\theta} - \sum_f w^f \frac{dn^{if,s}}{d\theta} = \sum_j \frac{dp^j}{d\theta} \left(\bar{y}^{ij,s} - c^{ij} \right) \\
+ \sum_f \frac{dw^f}{d\theta} \left(n^{if,s} + \bar{n}^{if,s} \right) + \sum_j \nu^{ij} \frac{d\pi^j}{d\theta}
\end{aligned}
$$

Proof #3 of First Welfare Theorem III

- After aggregating across all individuals, it must be that

$$\sum_i \frac{\frac{dV^i}{d\theta}}{\lambda^i} = \sum_j \frac{dp^j}{d\theta} \sum_i \left(\bar{y}^{ij,s} - c^{ij}\right) + \sum_f \frac{dw^f}{d\theta} \sum_i \left(n^{if,s} + \bar{n}^{if,s}\right) + \sum_j \overbrace{\sum_i \nu^{ij}}^{=1} \frac{d\pi^j}{d\theta}$$

- So

$$\sum_i \frac{\frac{dV^i}{d\theta}}{\lambda^i} = \sum_j \frac{dp^j}{d\theta} \sum_i \left(\bar{y}^{ij,s} + y^{j,s} - c^{ij}\right) + \sum_f \frac{dw^f}{d\theta} \sum_i \left(n^{if,s} + \bar{n}^{if,s} - n^{f,d}\right) = 0$$

Proof #3 of First Welfare Theorem IV

- The final argument follows again by contradiction. Since $\sum_i \frac{\frac{dV^i}{d\theta}}{\lambda^i} = 0$, if for some individual $\frac{\frac{dV^i}{d\theta}}{\lambda^i} > 0$, there must be another individual for whom $\frac{\frac{dV^i}{d\theta}}{\lambda^i} < 0$, so every perturbation features losers, implying that the competitive equilibrium is Pareto efficient.

References I

Dávila, E., and A. Schaab (2024): "Welfare Accounting," *Working Paper*.