ECON 500a
General Equilibrium and Welfare Economics
Dynamic Economies

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Outline: Dynamic Stochastic Economies

1. Dynamic Economies
2. Stochastic Economies
3. Asset Pricing
4. Efficiency and Welfare
5. Incomplete Markets
6. Production, Firms, Ownership

- Readings
  - MWG: Chapter 20, 19.B, 19.C, 19.D,

Defining Goods/Commodities

- What is a good/factor/commodity?

  goods (services, factors) consumed, produced, exchanged, or supplied at different times, states, or locations are different goods

1. **Time**: an apple today is a different good from the same apple tomorrow
2. **State**: an apple when sunny is a different good from the same apple when rainy
3. **Location**: an apple hanging from the tree is a different good from the same apple at the store.
   - Production economies nest location → expanding the set $\mathcal{J}$
   - e.g. good 1 and good 2 could be the "same" good in different locations → these are really different goods
   - e.g. Iceberg costs in Armington model

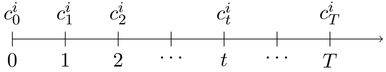
- Dynamic stochastic economies are very different

Modeling Time

- Discrete time
- Countable dates, indexed by $t \in \{0, 1, \ldots, T\}$, where $T \leq \infty$
  Similar with continuous time
  - A date is a point in time in which economic activity (consumption, exchange, production) takes place
- $T < \infty$: finite-horizon $\rightarrow$ this course
- $T = \infty$: infinite-horizon $\rightarrow$ macro
- No uncertainty $\rightarrow$ perfect foresight

Intertemporal Preferences

- Often single-good economies
- Timeline



- General preferences

$$V^i = u^i\left(c_0^i, \ldots, c_t^i, \ldots, c_T^i\right)$$

  - c_t^i: consumption of individual i at date t
  - Consistent notation $\rightarrow$ t as subscript

Intertemporal Preferences

- Standard intertemporal preferences
  Backus, Routledge, and Zin (2005): "exotic" preferences

$$V^i = \sum_{t=0}^{T} \left(\beta^i\right)^t u^i\left(c_t^i\right)$$

  - Time separable + Exponential discounting
  - Why? Recursive + time consistent: $V_t^i = u^i\left(c_t^i\right) + \beta^i V_{t+1}^i$
  - β^i: discount factor
  - $u^i(\cdot)$: instantaneous/flow utility
    - Inada condition: $\lim_{c_t^i \to 0} \frac{\partial u^i}{\partial c_t^i} = \infty$

Intertemporal Preferences

- Typically → isoelastic preferences: $u^i\left(c_t^i\right) = \frac{\left(c_t^i\right)^{1-\frac{1}{\psi}}}{1-\frac{1}{\psi}}$
  - ψ is EIS (elasticity of intertemporal substitution)
  - $\psi \to 1$: log utility
  - Macro → $\psi = \frac{1}{2}$
  - Finance → $\psi = 1.5$ (Epstein-Zin)
- With multiple goods and factors:

$$V^i = \sum_{t=0}^{T} \left(\beta^i\right)^t u^i \left(\left\{ c_t^{ij} \right\}_{j \in \mathcal{J}}, \left\{ n_t^{if,s} \right\}_{f \in \mathcal{F}} \right)$$

Fisher Economy

- Fisher Economy ⇒ How are interest rates determined?
  Fisher (1930)
- Single good endowment economy
  - $I \geq 1$, $J = 1$, $T = 1$
  - If $I = 2$, Fisher economy becomes a static Edgeworth box economy
  - $T = 1$ means two dates: $t = 0$ and $t = 1$
- Roadmap
  1. Physical Structure
  2. Once-and-for-all Trading
  3. Sequential Trading

Fisher Economy: Physical Structure

- Preferences

$$V^i = u^i\left(c_0^i\right) + \beta^i u^i\left(c_1^i\right)$$

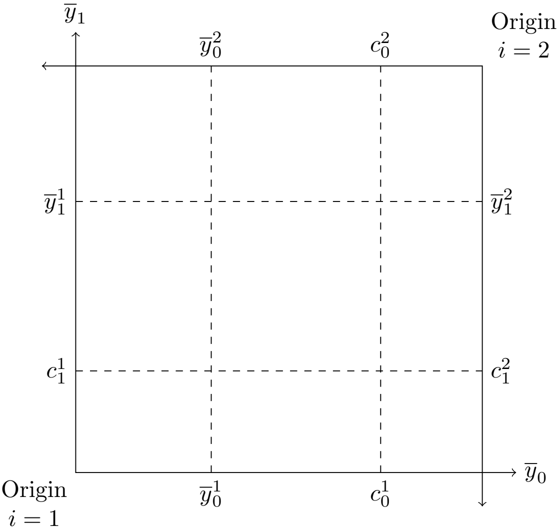
- Resource constraints

$$\sum_i c_t^i = \sum_i \bar{y}_t^i$$

- If $I = 2$:

$$\begin{aligned} V^1 &= u^1\left(c_0^1\right) + \beta^1 u^1\left(c_1^1\right) && \text{Preferences Individual 1} \\ V^2 &= u^2\left(c_0^2\right) + \beta^2 u^2\left(c_1^2\right) && \text{Preferences Individual 2} \\ c_0^1 + c_0^2 &= \bar{y}_0^1 + \bar{y}_0^2 && \text{Resource Constraint Date 0} \\ c_1^1 + c_1^2 &= \bar{y}_1^1 + \bar{y}_1^2 && \text{Resource Constraint Date 1} \end{aligned}$$

Fisher Economy: Box Diagram



Fisher Economy: Planning Problem

$$\max_{\{c_t^i\}} \sum_i \alpha^i \sum_t \left(\beta^i\right)^t u^i\left(c_t^i\right), \qquad \text{s.t.} \qquad \sum_i c_t^i = \sum_i \bar{y}_t^i, \quad \forall t$$

- Same as static exchange economy!
- Lagrangian

$$\mathcal{L} = \sum_i \alpha^i \sum_t \left(\beta^i\right)^t u^i\left(c_t^i\right) - \sum_t \eta_t \left(\sum_i c_t^i - \sum_i \bar{y}_t^i\right)$$

- Optimality conditions

$$\frac{d\mathcal{L}}{dc_t^i} = \alpha^i \left(\beta^i\right)^t \frac{\partial u^i}{\partial c_t^i} - \eta_t = 0$$

  - Efficiency

$$\frac{\beta^i \frac{\partial u^i}{\partial c_1^i}}{\frac{\partial u^i}{\partial c_0^i}} = \frac{\eta_1}{\eta_0}, \ \forall i$$

  - Redistribution

$$\frac{\frac{\partial u^i}{\partial c_0^i}}{\frac{\partial u^n}{\partial c_0^n}} = \frac{\beta^i \frac{\partial u^i}{\partial c_1^i}}{\beta^i \frac{\partial u^n}{\partial c_1^n}} = \frac{\alpha^n}{\alpha^i}$$

Competitive Equilibrium: Once-and-for-all Trading

- Dynamic economies ⇒ More subtle equilibrium definitions
  1. Once-and-for-all Trading ⇒ Arrow-Debreu
  2. Sequential Trading ⇒ Radner
- Once-and-for-all
  - Consumption at each date has linear price p_t
  - Individual initially trade and commit to deliver as time goes by
  - Delivery of goods occurs sequentially as time unfolds
  - Full commitment to deliver goods: infinite punishments

Competitive Equilibrium: Once-and-for-all Trading

A competitive equilibrium is an allocation $\{c_t^i\}$, and prices $\{p_t\}$, such that

i) individuals choose consumption paths to maximize utility subject to their budget constraint taking prices as given, that is, they solve

$$\max_{c_t^i} \sum_t \left(\beta^i\right)^t u^i\left(c_t^i\right) \quad \text{s.t.} \quad \sum_t p_t c_t^i = \sum_t p_t \bar{y}_t^i, \quad \forall i,$$

ii) and markets clear, that is, resource constraints hold:

$$\sum_i c_t^i = \sum_i \bar{y}_t^i, \quad \forall t.$$

- Lagrangian, assuming interior solution:

$$\mathcal{L} = \sum_t \left(\beta^i\right)^t u^i\left(c_t^i\right) - \lambda^i \left(\sum_t p_t c_t^i - \sum_t p_t \bar{y}_t^i\right)$$

- Optimality conditions

$$\frac{d\mathcal{L}}{dc_t^i} = \left(\beta^i\right)^t \frac{\partial u^i}{\partial c_t^i} - \lambda^i p_t = 0$$

Competitive Equilibrium: Once-and-for-all Trading

- When $T = 1$ case, these conditions imply

$$\frac{\beta^i \frac{\partial u^i}{\partial c_1^i}}{\frac{\partial u^i}{\partial c_0^i}} = \frac{p_1}{p_0} \quad (1)$$

  - RHS is rate at which i can exchange consumption at date 0 vs. date 1 — note that $\dim\left(\frac{p_1}{p_0}\right) = \frac{\text{consumption at date 0}}{\text{consumption at date 1}}$ — at the market in which all trades take place
  - Optimal to equalize the rate of exchange to the individual's marginal rate of substitution between consumption at both dates.
- $\frac{p_0}{p_1}$ is the gross *interest rate*, and is typically written as

$$\frac{p_0}{p_1} \equiv 1 + r_1$$

  - $1 + r_1$ is the price of date 0 consumption in terms of date 1 consumption (interest rate)
  - $\frac{1}{1+r_1}$ is the price of date 1 consumption in terms of date 0 consumption (discount factor)
  - Interest rate is real, since it is expressed in units of consumption

Competitive Equilibrium: Once-and-for-all Trading

- *Present value* and *future value* budget constraints:

$$\text{(Present Value)} \qquad c_0^i + \frac{1}{1+r_1} c_1^i = \bar{y}_0^i + \frac{1}{1+r_1} \bar{y}_1^i$$

$$\text{(Future Value)} \qquad (1+r_1)\, c_0^i + c_1^i = (1+r_1)\, \bar{y}_0^i + \bar{y}_1^i$$

  - Just choosing numeraire!
- Individual optimality is an *Euler equation:*

$$\frac{\partial u^i}{\partial c_0^i} = (1+r_1)\, \beta^i \frac{\partial u^i}{\partial c_1^i}$$

  - Individual must be indifferent between
    - consuming at date 0, which yields a utility gain of $\frac{\partial u^i}{\partial c_0^i}$
    - using that unit of date-0 consumption to purchase $\frac{p_0}{p_1} \equiv 1 + r_1$ units of date 1 consumption, which when consumed yield a utility gain of $\beta^i \frac{\partial u^i}{\partial c_1^i}$

Competitive Equilibrium: Sequential Trading

- More realistic $\Rightarrow$ Trade as time unfolds
  - Trade occurs *sequentially*, under the assumption that individuals cannot renege on their promises
- Definition: an *asset* is a tradable claim that allows individuals to exchange resources across dates (and, in stochastic economies, states)
- One asset in this economy: *risk-free* or *riskless asset*

Competitive Equilibrium: Sequential Trading

A competitive equilibrium is a consumption allocation $\{c_t^i\}$, an asset allocation $\{a_t^i\}$, and a path of interest rates $\{1+r_t\}$, such that

i) individuals choose consumption paths to maximize utility subject to their budget constraint taking prices as given, that is, they solve

$$\max_{\{c_t^i, a_t^i\}} \sum_t \left(\beta^i\right)^t u^i\left(c_t^i\right) \quad \text{s.t.} \quad c_t^i + a_t^i = \bar{y}_t^i + (1+r_t)\, a_{t-1}^i, \ \forall t, \ \forall i,$$

where $a_{-1}^i = a_T^i = 0, \ \forall i$
we could have > 0, but aggregates should still be zero

ii) and markets clear at each date, that is, resource constraints hold:

$$\sum_i c_t^i = \sum_i \bar{y}_t^i, \quad \forall t,$$

and the asset market clears at each date:

$$\sum_i a_t^i = 0, \quad \forall t < T.$$

- If $a_0^i > 0 \Rightarrow$ Saving
- If $a_0^i < 0 \Rightarrow$ Borrowing
- # of budget constraints: $T+1$ vs. 1

Competitive Equilibrium: Sequential Trading

- If $I = 2$

$$\max_{c_0^i, c_1^i, a_0^i} u^i\left(c_0^i\right) + \beta^i u^i\left(c_1^i\right) \quad \text{s.t.}$$

$$\begin{aligned} c_0^i + a_0^i &= \bar{y}_0^i \\ c_1^i &= \bar{y}_1^i + (1 + r_1)\, a_0^i \end{aligned}$$

- And

$$\begin{aligned} c_0^1 + c_0^2 &= \bar{y}_0^1 + \bar{y}_0^2 && \text{Market Clearing Date 0} \\ c_1^1 + c_1^2 &= \bar{y}_1^1 + \bar{y}_1^2 && \text{Market Clearing Date 1} \end{aligned}$$

$$\sum_i a_0^i = 0 \qquad \text{Asset Market Clearing}$$

Competitive Equilibrium: Sequential Trading

- Lagrangian assuming an interior solution:

$$\mathcal{L} = \sum_t \left(\beta^i\right)^t \left(u^i\left(c_t^i\right) - \lambda_t^i\left(c_t^i + a_t^i - \bar{y}_t^i - \left(1 + r_t\right) a_{t-1}^i\right)\right)$$

- Optimality conditions

$$\frac{d\mathcal{L}}{dc_t^i} = \frac{\partial u^i}{\partial c_t^i} - \lambda_t^i = 0 \text{ and } \frac{d\mathcal{L}}{da_t^i} = -\left(\beta^i\right)^t \lambda_t^i + \left(\beta^i\right)^{t+1}\left(1 + r_{t+1}\right)\lambda_{t+1}^i = 0$$

  - "All Hamiltonians are Lagrangians"
- $T = 1 \Rightarrow$ *Euler equation:*

$$\lambda_t^i = \beta^i\left(1 + r_{t+1}\right)\lambda_{t+1}^i \Rightarrow \frac{\partial u^i}{\partial c_0^i} = \left(1 + r_1\right)\beta^i \frac{\partial u^i}{\partial c_1^i}$$

which implies that

$$\frac{\beta^i \frac{\partial u^i}{\partial c_1^i}}{\frac{\partial u^i}{\partial c_0^i}} = \frac{1}{1 + r_1}$$

- Same as once-and-for-all trading!

Competitive Equilibrium: Equivalence

- Perfect foresight + one asset = Complete Markets

$$\text{Once-and-for-all Trade} \underbrace{\Longleftrightarrow}_{\substack{\text{Complete} \\ \text{Markets}}} \text{Sequential Trade}$$

- Proof: consolidate budget constraints
  - $T = 1$ case

$$a_0^i = \frac{c_1^i - \bar{y}_1^i}{1 + r_1} \Rightarrow c_0^i + \frac{c_1^i - \bar{y}_1^i}{1 + r_1} = \bar{y}_0^i \Rightarrow c_0^i + \frac{c_1^i}{1 + r_1} = \bar{y}_0^i + \frac{\bar{y}_1^i}{1 + r_1}$$

- $T > 1 \Rightarrow$ recursive substitution
- Why is this equivalence result important? We can apply all the result from Block I!

Walras' Law: Once-and-for-all

1. **Once-and-for-all**: same as static exchange

- Aggregating budget constraints

$$\sum_t p_t \left(\sum_i c_t^i - \sum_i \bar{y}_t^i \right) = 0$$

- If $\sum_i c_t^i = \sum_i y_t^i$ at all but one date, the market for consumption at that one date must also clear

Walras' Law: Sequential Trading

2. **Sequential trading**

- Aggregating all budget constraints at date t

$$\sum_i c_t^i + \sum_i a_t^i = \sum_i \bar{y}_t^i + (1 + r_t) \sum_i a_{t-1}^i \Rightarrow a_t = \bar{y}_t - c_t + (1 + r_t)\, a_{t-1},$$

  where $a_t = \sum_i a_t^i$, $c_t = \sum_i a_t^i$, and $\bar{y}_t = \sum_i \bar{y}_t^i$

- If asset markets clear at all dates, $a_t = 0$, then consumption markets must clear too:

$$a_t = 0,\ \forall t \Rightarrow \bar{y}_t - c_t = 0,$$

- If consumption markets clear at all dates, $\bar{y}_t = c_t$, then asset markets must also clear

$$\bar{y}_t - c_t = 0,\ \forall t \Rightarrow a_t = 0,$$

  where we used the fact that $a_{t-1} = 0$

Competitive Equilibrium: Extensions

1. Permanent Income Hypothesis

$$c_0^i = \frac{1}{1+\beta^{\psi}\left(1+r_1\right)^{\psi-1}} \underbrace{\left(\bar{y}_0^i + \frac{1}{1+r_1}\bar{y}_1^i\right)}_{=\text{Permanent Income}}$$

$$c_1^i = \frac{\beta^{\psi}\left(1+r_1\right)^{\psi-1}}{1+\beta^{\psi}\left(1+r_1\right)^{\psi-1}} \underbrace{\left(\bar{y}_0^i + \frac{1}{1+r_1}\bar{y}_1^i\right)}_{=\text{Permanent Income}}$$

2. Multiple Goods: Physical Structure

$$V^i = \sum_t \left(\beta^i\right)^t u^i\left(\left\{c_t^{ij}\right\}_{j\in\mathcal{J}}\right)$$

$$\sum_i c_t^{ij} = \sum_i \bar{y}_t^{ij}$$

Competitive Equilibrium: Extensions

- Multiple Goods: Once-and-for-all trading

$$\max_{c_t^{ij}} \sum_t \left(\beta^i\right)^t u^i \left(\left\{ c_t^{ij} \right\}_{j \in \mathcal{J}} \right) \quad \text{s.t.} \quad \sum_t \sum_j p_t^j c_t^{ij} = \sum_t \sum_j p_t^j \bar{y}_t^{ij}$$

  - Resource constraints: $\sum_i c_t^{ij} = \sum_i \bar{y}_t^{ij}$

Competitive Equilibrium: Extensions

- Multiple Goods: Sequential trading

$$\max_{\left\{c_t^{ij}, a_t^i\right\}} \sum_t \left(\beta^i\right)^t u^i \left(\left\{c_t^{ij}\right\}_{j \in \mathcal{J}}\right) \quad \text{s.t.}$$

$$\sum_j p_t^j c_t^{ij} + a_t^i = \sum_j p_t^j \bar{y}_t^{ij} + (1 + r_t)\, a_{t-1}^i, \ \forall t, \ \forall i$$

  where $a_{-1}^i = a_T^i = 0$, $\forall i$,
  - Resource constraints: $\sum_i c_t^{ij} = \sum_i \bar{y}_t^{ij}, \quad \forall j, \forall t,$
  - Asset market clearing: $\sum_i a_t^i = 0, \quad \forall t < T$
  - Assumption: financial asset in dollars
    - Alternative: financial asset pays in real terms (e.g. in apples) (Geanakoplos and Mas-Colell, 1989)
- $T + 1$ normalizations: one p_t^j per period
- Alternative numeraire/asset denomination
  - Choose good 1 as numeraire at each date ($p_t^1 = 1$)
  - Define real asset in units of the numeraire (good 1)

$$c_t^{i1} + \sum_{j=2}^{J} p_t^j c_t^{ij} + a_t^i = \bar{y}_t^{i1} + \sum_{j=2}^{J} p_t^j \bar{y}_t^{ij} + (1 + r_t)\, a_{t-1}^i$$

References I

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