ECON 500a
General Equilibrium and Welfare Economics
Stochastic Economies

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Outline: Dynamic Stochastic Economies

1. Dynamic Economics
2. Stochastic Economics
3. Asset Pricing
4. Efficiency and Welfare
5. Incomplete Markets
6. Production, Firms, Ownership

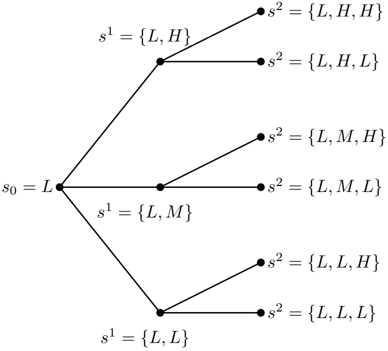
- Readings
  - MWG: Chapter 19

Modeling Uncertainty

- History notation
  - Dates: $t \in \{0, \dots, T\}$, where $0 \leq T \leq \infty$
  - Events: $s_t \in \mathcal{S}_t$
  - History: $s^t = (s_0, s_1, \dots, s_t)$, with probability $\pi_t\left(s^t\right)$
    - Sustained assumption: common beliefs
    - We could have $\pi_t^i\left(s^t\right)$ (heterogeneous beliefs)
  - Initial event/history $s_0 = s^0$ is predetermined, so $\pi_0\left(s_0\right) = 1$
    This is without loss of generality

Example

- Suppose $\mathcal{S}_0 = \{L\}$, $\mathcal{S}_1 = \{H, M, L\}$ and $\mathcal{S}_2 = \{H, L\}$
- Possible states (s_t) and histories (s^t) at $t = 1$
  - $s_1 = L$ and $s^1 = \{L, L\}$
  - $s_1 = M$ and $s^1 = \{L, M\}$
  - $s_1 = H$ and $s^1 = \{L, H\}$
- Possible states and histories at $t = 2$ if $s_1 = M$
  Same with $s_1 = H$ or $s_1 = L$
  - $s_2 = L$ and $s^2 = \{L, M, L\}$
  - $s_2 = H$ and $s^2 = \{L, M, H\}$



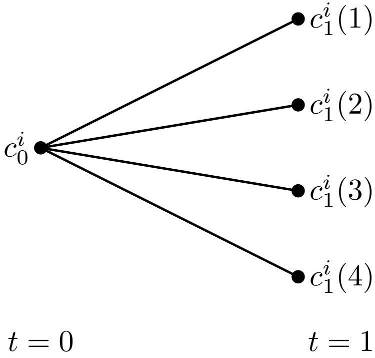
Modeling Uncertainty

- Problem with histories
  - Curse of dimensionality: exponential growth with t
- Solutions
  - Recursive methods $\Rightarrow$ Ljungqvist and Sargent (2018)
  - Binomial economies $\Rightarrow$ Recombining trees $\Rightarrow$ Brownian motion
- We focus on $T = 1$ (at most $T = 2$) $\Rightarrow$ Not a problem for us
- Equivalent formulation: filtrations
  - Good for information, not so good for intermediate consumption

Modeling Uncertainty

- If $T = 1$: simpler notation
  - Drop s_0
  - Events/States/histories at $t = 1$: $s \in \{1, \ldots, S\}$
- Consumption

$$c_0^i \quad \text{and} \quad c_1^i(1), \ldots c_1^i(s), \ldots, c_1^i(S).$$



Risk Preferences

- Often single-good economies
- General preferences

$$V^i = u^i\left(c_0^i\left(s^0\right), \ldots, \left\{c_t^i\left(s^t\right)\right\}, \ldots, \left\{c_T^i\left(s^T\right)\right\}; \left\{\pi_t\left(s^t\right)\right\}\right)$$

- Standard preferences: *expected utility* ⇒ "independence axiom"
  Backus, Routledge, and Zin (2005): "exotic" preferences

$$V^i = \sum_t \left(\beta^i\right)^t \sum_{s^t} \pi_t\left(s^t\right) u^i\left(c_t^i\left(s^t\right)\right)$$

  - Again: time separable + exponential discounting
- $T = 1$ case:

$$\boxed{V^i = u^i\left(c_0^i\right) + \beta^i \sum_s \pi_1(s)\, u^i\left(c_1^i(s)\right)}$$

- Many goods and factors:

$$V^i = \sum_t \left(\beta^i\right)^t \sum_{s^t} \pi_t\left(s^t\right) u^i\left(\left\{c_t^{ij}\left(s^t\right)\right\}_{j\in\mathcal{J}}, \left\{n_t^{if,s}\left(s^t\right)\right\}_{f\in\mathcal{F}}\right)$$

Risk Preferences

- Typically → CRRA preferences: $u^i\left(c_t^i\left(s^t\right)\right) = \frac{\left(c_t^i\left(s^t\right)\right)^{1-\gamma}}{1-\gamma}$
  - $\gamma = -c_t^i\left(s^t\right)\frac{u^{i\prime\prime}\left(c_t^i\left(s^t\right)\right)}{u^{i\prime}\left(c_t^i\left(s^t\right)\right)}$ is coefficient of relative risk aversion
  - $\gamma \rightarrow 1$: log utility
  - Macro → $\gamma = \frac{1}{2}$
  - Finance → $\gamma = 10$ (Epstein-Zin)
- Also CARA preferences: $u^i\left(c_t^i\left(s^t\right)\right) = -e^{-\gamma c_t^i\left(s^t\right)}$
  - $\gamma = -\frac{u^{i\prime\prime}\left(c_t^i\left(s^t\right)\right)}{u^{i\prime}\left(c_t^i\left(s^t\right)\right)}$ is coefficient of absolute risk aversion
- Epstein-Zin with $T = 1$: $\beta^i = \frac{\hat{\beta}^i}{1-\hat{\beta}^i}$

$$V^i = \left[\left(1-\hat{\beta}^i\right)\left(c_0^i\left(s_0\right)\right)^{1-\frac{1}{\psi}} + \hat{\beta}^i\left(\sum_{s^1}\pi_1\left(s^1\right)\left[\left(c_1^i\left(s^1\right)\right)^{1-\gamma}\right]\right)^{\frac{1-\frac{1}{\psi}}{1-\gamma}}\right]^{\frac{1}{1-\frac{1}{\psi}}}$$

  - Recursive preferences
  - Early ($\gamma > \frac{1}{\psi}$) vs late resolution of uncertainty ($\gamma < \frac{1}{\psi}$)

Finance Economy: Roadmap

Finance Economy

1. Physical Structure
2. Planning Problem
3. Once-and-for-all Trading
4. Sequential Trading with Arrow-Debreu Securities
5. Sequential Trading with General Asset Structure

Physical Structure

- Finance Economy $\Rightarrow$ asset pricing, corporate finance, representative agent macro, heterogeneous agents macro
- Single good endowment economy: $I \geq 1$, $J = 1$, $T = 1$, $S \geq 1$
- Preferences

$$V^i = u^i\left(c_0^i\right) + \beta^i \sum_s \pi_1(s)\, u^i\left(c_1^i(s)\right)$$

- Resource constraints

$$\sum_i c_0^i = \sum_i \bar{y}_0^i \qquad \text{and} \qquad \sum_i c_1^i(s) = \sum_i \bar{y}_1^i(s)\,,\ \forall s$$

Physical Structure

- $I = S = 2$

$$
\begin{aligned}
V^1 &= u^1\left(c_0^1\right) + \beta^1 u^1\left(c_1^1(1)\right) + \beta^1 u^1\left(c_1^1(2)\right)\\
V^2 &= u^2\left(c_0^2\right) + \beta^2 u^2\left(c_1^2(1)\right) + \beta^2 u^2\left(c_1^2(2)\right)\\
c_0^1 + c_0^2 &= \bar{y}_0^1 + \bar{y}_0^2\\
c_1^1(1) + c_1^2(1) &= \bar{y}_1^1(1) + \bar{y}_1^2(1)\\
c_1^1(2) + c_1^2(2) &= \bar{y}_1^1(2) + \bar{y}_1^2(2)
\end{aligned}
$$

Planning Problem

$$\max_{\{c_0^i, c_1^i(s)\}} \sum_i \alpha^i \left(u^i\left(c_0^i\right) + \beta^i \sum_s \pi_1(s)\, u^i\left(c_1^i(s)\right) \right), \quad \text{s.t.}$$

$$\sum_i c_0^i = \sum_i \bar{y}_0^i \qquad \text{and} \qquad \sum_i c_1^i(s) = \sum_i \bar{y}_1^i(s), \ \forall s$$

- Lagrangian

$$\mathcal{L} = \sum_i \alpha^i \left(u^i\left(c_0^i\right) + \beta^i \sum_s \pi_1(s)\, u^i\left(c_1^i(s)\right) \right)$$
$$- \eta_0 \left(\sum_i c_0^i - \sum_i \bar{y}_0^i \right) - \sum_s \eta_1(s) \left(\sum_i c_1^i(s) - \sum_i \bar{y}_1^i(s) \right)$$

- Optimality conditions

$$\frac{d\mathcal{L}}{dc_0^i} = \alpha^i \frac{\partial u^i}{\partial c_0^i} - \eta_0 = 0$$
$$\frac{d\mathcal{L}}{dc_1^i(s)} = \alpha^i \beta^i \pi_1(s) \frac{\partial u^i}{\partial c_1^i(s)} - \eta_1(s) = 0$$

Planning Problem

- Efficient allocation across states:

$$\frac{\frac{\partial u^i}{\partial c_1^i(s)}}{\frac{\partial u^i}{\partial c_1^i(s')}} = \frac{\eta_1(s)}{\eta_1(s')}, \ \forall i$$

- Efficient intertemporal allocation:

$$\frac{\beta^i \pi_1(s) \frac{\partial u^i}{\partial c_1^i(s)}}{\frac{\partial u^i}{\partial c_0^i}} = \frac{\eta_1(s)}{\eta_0} \ \Rightarrow \ \frac{\sum_s \beta^i \pi_1(s) \frac{\partial u^i}{\partial c_1^i(s)}}{\frac{\partial u^i}{\partial c_0^i}} = \frac{\sum_s \eta_1(s)}{\eta_0}, \ \forall i$$

- Redistribution:

$$\frac{\frac{\partial u^i}{\partial c_0^i}}{\frac{\partial u^n}{\partial c_0^n}} = \frac{\sum_s \beta^i \pi_1(s) \frac{\partial u^i}{\partial c_1^i(s)}}{\sum_s \beta^n \pi_1(s) \frac{\partial u^n}{\partial c_1^n(s)}} = \frac{\alpha^n}{\alpha^i}$$

for individuals i and n

Once-and-for-all Trading

A *competitive equilibrium* is an allocation $\left\{c_0^i, c_1^i(s)\right\}$, and prices $\{p_0, p_1(s)\}$, such that

i) individuals choose consumption paths to maximize utility subject to their budget constraint taking prices as given, that is, they solve

$$\max_{c_t^i} u^i\left(c_0^i\right) + \beta^i \sum_s \pi_1(s)\, u^i\left(c_1^i(s)\right) \quad \text{s.t.}$$

$$p_0 c_0^i + \sum_s p_1(s)\, c_1^i(s) = p_0 \bar{y}_0^i + \sum_s p_1(s)\, \bar{y}_1^i(s)$$

- p_0 is the price of consumption at date 0
- $p_1(s)$ is the price of consumption at date 1 in state s

ii) and markets clear, that is, resource constraints hold:

$$\sum_i c_0^i = \sum_i \bar{y}_0^i \qquad \text{and} \qquad \sum_i c_1^i(s) = \sum_i \bar{y}_1^i(s)\,,\ \forall s$$

- How many budget constraints do we have?

Once-and-for-all Trading

- Lagrangian:

$$\mathcal{L} = u^i\left(c_0^i\right) + \beta^i \sum_s \pi_1(s)\, u^i\left(c_1^i(s)\right)$$
$$- \lambda^i \left(p_0 c_0^i + \sum_s p_1(s)\, c_1^i(s) - p_0 \bar{y}_0^i - \sum_s p_1(s)\, \bar{y}_1^i(s)\right)$$

- Optimality conditions:

$$\frac{d\mathcal{L}}{dc_0^i} = \frac{\partial u^i}{\partial c_0^i} - \lambda^i p_0 = 0$$
$$\frac{d\mathcal{L}}{dc_1^i(s)} = \beta^i \pi_1(s)\, \frac{\partial u^i}{\partial c_1^i(s)} - \lambda^i p_1(s) = 0$$

Once-and-for-all Trading

- These conditions imply that

$$\boxed{\frac{\beta^i \pi_1(s) \frac{\partial u^i}{\partial c_1^i(s)}}{\frac{\partial u^i}{\partial c_0^i}} = \frac{p_1(s)}{p_0}} \quad \text{and} \quad \boxed{\frac{\pi_1(s) \frac{\partial u^i}{\partial c_1^i(s)}}{\pi_1(s') \frac{\partial u^i}{\partial c_1^i(s')}} = \frac{p_1(s)}{p_1(s')}}$$

- $\dim\left(\frac{p_1(s)}{p_0}\right) = \frac{\text{consumption at date 0}}{\text{consumption at state } s}$ and $\dim\left(\frac{p_1(s)}{p_1(s')}\right) = \frac{\text{consumption at date } s'}{\text{consumption at state } s}$
- CRRA preferences + common discount factor and beliefs:

$$\left(\frac{c_1^i(s)}{c_0^i}\right)^{-\gamma} \quad \text{and} \quad \left(\frac{c_1^i(s)}{c_1^i(s')}\right)^{-\gamma} \Rightarrow \text{ equalized across } i$$

  - Consumption comovement

Sequential Trading with Arrow-Debreu Securities

- The *Arrow-Debreu security* associated with state s is an asset that pays a unit of consumption at state s
  - The price of an Arrow-Debreu security is a *state-price*
- A *competitive equilibrium* is a consumption allocation $\left\{c_0^i, c_1^i(s)\right\}$, an asset allocation $\left\{a_0^i(s)\right\}$, and state-prices $\mu_0(s)$, such that
  i) individuals choose consumption to maximize utility subject to their budget constraint taking prices as given, that is, they solve

$$\max_{\left\{c_0^i c_1^i(s), a_0^i(s)\right\}} u^i\left(c_0^i\right) + \beta^i \sum_s \pi_1(s) u^i\left(c_1^i(s)\right) \quad \text{s.t.}$$

$$c_0^i + \sum_s \mu_0(s) a_0^i(s) = \bar{y}_0^i \qquad \text{and} \qquad c_1^i(s) = \bar{y}_1^i(s) + a_0^i(s), \ \forall s,$$

  ii) and markets clear, that is, resource constraints hold:

$$\sum_i c_0^i = \sum_i \bar{y}_0^i \qquad \text{and} \qquad \sum_i c_1^i(s) = \sum_i \bar{y}_1^i(s), \ \forall s,$$

  and the financial markets for Arrow-Debreu securities clear:

$$\sum_i a_0^i(s) = 0, \quad \forall s$$

- How many budget constraints do we have?

Sequential Trading with Arrow-Debreu Securities

- $I = S = 2$

$$\max_{c_0^i, c_1^i(1), c_1^i(2), a_0^1(1), a_0^1(2)} u^i\left(c_0^i\right) + \beta^i \pi_1(1) u^i\left(c_1^i(1)\right) + \beta^i \pi_1(2) u^i\left(c_1^i(2)\right) \quad \text{s.t.}$$

$$\begin{aligned} c_0^i + \mu_0(1) a_0^1(1) + \mu_0(2) a_0^1(2) &= \bar{y}_0^i \\ c_1^i(1) &= \bar{y}_1^i(1) + a_0^1(1) \\ c_1^i(2) &= \bar{y}_1^i(2) + a_0^1(2) \end{aligned}$$

- And

$$\begin{aligned} c_0^1 + c_0^2 &= \bar{y}_0^1 + \bar{y}_0^2 && \text{Market Clearing Date 0} \\ c_1^1(1) + c_1^2(1) &= \bar{y}_1^1(1) + \bar{y}_1^2(1) && \text{Market Clearing State 1} \\ c_1^1(2) + c_1^2(2) &= \bar{y}_1^1(2) + \bar{y}_1^2(2) && \text{Market Clearing State 2} \end{aligned}$$

$$a_0^1(1) + a_0^2(1) = 0 \text{ and } a_0^1(2) + a_0^2(2) = 0 \qquad \text{Asset Market Clearing}$$

Sequential Trading with Arrow-Debreu Securities

- Lagrangian

$$\mathcal{L} = u^i\left(c_0^i\right) + \beta^i \sum_s \pi_1(s)\, u^i\left(c_1^i(s)\right) - \lambda_0^i\left(c_0^i + \sum_s \mu_0(s)\, a_0^i(s) - \bar{y}_0^i\right)$$
$$- \sum_s \lambda_1^i(s)\left(c_1^i(s) - \bar{y}_1^i(s) - a_0^i(s)\right)$$

- Optimality conditions

$$\frac{d\mathcal{L}}{dc_0^i} = \frac{\partial u^i}{\partial c_0^i} - \lambda_0^i = 0$$
$$\frac{d\mathcal{L}}{dc_1^i(s)} = \beta^i \pi_1(s)\frac{\partial u^i}{\partial c_1^i(s)} - \lambda_1^i(s) = 0$$
$$\frac{d\mathcal{L}}{da_0^i(s)} = -\mu_0(s)\,\lambda_0^i + \lambda_1^i(s) = 0 \quad \text{(Euler Equation)}$$

- Therefore

$$\boxed{\frac{\beta^i \pi_1(s)\frac{\partial u^i}{\partial c_1^i(s)}}{\frac{\partial u^i}{\partial c_0^i}} = \mu_0(s)} \quad \text{and} \quad \boxed{\frac{\pi_1(s)\frac{\partial u^i}{\partial c_1^i(s)}}{\pi_1(s')\frac{\partial u^i}{\partial c_1^i(s')}} = \frac{\mu_0(s)}{\mu_0(s')}}$$

- Left: LHS is individual MRS, RHS is state-price
- Right: LHS is individual MRS, RHS is ratio of state-prices

Equivalence

- Boxed equations in once-and-for-all and sequential trading with Arrow-Debreu securities are equivalent
- Alternative $\Rightarrow$ budget constraint consolidation
  - Use date-1 budget constraint to solve for $a_0^i(s)$:

$$a_0^i(s) = c_1^i(s) - \bar{y}_1^i(s)$$

  - Substitute $a_0^i(s)$ into date-0 budget constraint:

$$c_0^i + \sum_s \mu_0(s)\, c_1^i(s) = \bar{y}_0^i + \sum_s \mu_0(s)\, \bar{y}_1^i(s)$$

  - This expression is equivalent to

$$c_0^i + \sum_s p_1(s)\, c_1^i(s) = \bar{y}_0^i + \sum_s p_1(s)\, \bar{y}_1^i(s)$$

  - State-price $\mu_0(s)$ plays the role of the once-and-for-all price $p_1(s)$
- Once-and-for-all trading equivalent to sequential trading with Arrow-Debreu securities

Sequential Trading with General Asset Structure

- General asset structures $\Rightarrow Z$ assets/securities in the economy
  - Assets indexed by $z \in \mathcal{Z} = \{1, \ldots, Z\}$
  - Payoff of asset z in state s at date 1 is $d_1^z(s) \geq 0$ units of consumption
  - Payoff vector of asset z: $\boldsymbol{d}_1^z = \left(d_1^z(1), \ldots, d_1^z(s), \ldots, d_1^z(S)\right)$
    Summary of asset z's payoffs over all states
  - q_0^z: price of asset z at date 0
- A *portfolio* of assets is given by a Z-dimensional vector of asset-holdings (portfolio positions):

$$\boldsymbol{a}_0^i = \begin{pmatrix} a_0^{i1} \\ \vdots \\ a_0^{iz} \\ \vdots \\ a_0^{iZ} \end{pmatrix}_{Z \times 1}$$

  - If $a_0^{iz} > 0$, individual i purchases asset z
  - If $a_0^{iz} < 0$, individual i short-sells, borrows, or issues asset z

Examples of Assets

i) The *risk-free* or *riskless* asset pays a unit of consumption in all states at a given date. Its the payoff vector is

$$(1, 1, 1, 1)$$

ii) An *Arrow-Debreu* security pays a unit of consumption at a particular state. Its payoff vector (sat for state $s = 3$) is

$$(0, 0, 1, 0)$$

- AD-securities form a "basis"

iii) A *stock* pays a varying amount of consumption, which without loss here can be increasing on the state. A possible payoff vector could be

$$(1, 2, 3, 4)$$

iv) A *call option* with strike K written on the stock has payoff: $C(K) = \max\{M - K, 0\}$. If $K = 2$ and $M = (1, 2, 3, 4)$, then

$$C(K) = (0, 0, 1, 2)$$

v) A *put option* with strike K written on the stock has payoff: $P(K) = \max\{K - M, 0\}$. If $K = 2$ and $M = (1, 2, 3, 4)$, then

$$P(K) = (1, 0, 0, 0)$$

Sequential Trading with General Asset Structure

- A *competitive equilibrium* is a consumption allocation $\left\{c_0^i, c_1^i(s)\right\}$, an asset allocation $\left\{a_0^{iz}\right\}$, and asset prices $\{q_0^z\}$, such that
  - i) individuals choose consumption to maximize utility subject to their budget constraint taking prices as given, that is, they solve

$$\max_{\left\{c_0^i, c_1^i(s), \left\{a_0^{iz}\right\}_{z\in\mathcal{Z}}\right\}} u^i\left(c_0^i\right) + \beta^i \sum_s \pi_1(s)\, u^i\left(c_1^i(s)\right) \quad \text{s.t.}$$

$$c_0^i + \sum_z q_0^z a_0^{iz} = \bar{y}_0^i \qquad \text{and} \qquad c_1^i(s) = \bar{y}_1^i(s) + \sum_z d_1^z(s)\, a_0^{iz},\ \forall s,$$

  - ii) and markets clear, that is, resource constraints hold:

$$\sum_i c_0^i = \sum_i \bar{y}_0^i \qquad \text{and} \qquad \sum_i c_1^i(s) = \sum_i \bar{y}_1^i(s),\ \forall s,$$

    and all asset markets clear:

$$\sum_i a_0^{iz} = 0, \quad \forall z$$

- How many budget constraints do we have?

Complete vs. Incomplete Markets

- Rewriting budget constraints as

$$c_1^i(s) - \bar{y}_1^i(s) = \sum_z d_1^z(s)\, a_0^{iz} \Rightarrow \boldsymbol{c}_1^i - \bar{\boldsymbol{y}}_1^i = \boldsymbol{D}\boldsymbol{a}_0^i,$$

where

$$\boldsymbol{c}_1^i = \begin{pmatrix} c_1^i(1) \\ \vdots \\ c_1^i(s) \\ \vdots \\ c_1^i(S) \end{pmatrix}_{S\times 1}, \quad \bar{\boldsymbol{y}}_1^i = \begin{pmatrix} \bar{y}_1^i(1) \\ \vdots \\ \bar{y}_1^i(s) \\ \vdots \\ \bar{y}_1^i(S) \end{pmatrix}_{S\times 1}$$

$$\boldsymbol{D} = \begin{pmatrix} \overbrace{d_1^1(1)}^{\text{asset } 1} & & \cdots & & \overbrace{d_1^z(1)}^{\text{asset}\, z} \\ & \ddots & & & \\ \vdots & & d_1^z(s) & & \vdots \\ & & & \ddots & \\ d_1^1(S) & & \cdots & & d_1^z(S) \end{pmatrix}_{S\times Z}, \quad \boldsymbol{a}_0^i = \begin{pmatrix} a_0^{i1} \\ \vdots \\ a_0^{iz} \\ \vdots \\ a_0^{iZ} \end{pmatrix}_{Z\times 1}$$

- $\boldsymbol{D}$ is the *payoff* matrix → dimension $S \times Z$ (# states × # assets)

Complete vs. Incomplete Markets

- An economy features *complete markets* if the available assets span all S states, that is, if the there are S assets with linearly independent returns or, equivalently, if

$$\text{rank}\,(\boldsymbol{D}) = S$$

  - Otherwise, and economy features *incomplete markets*.
- $Z < S$: fewer assets than states $\Rightarrow$ incomplete
- $Z = S$: same assets as states $\Rightarrow$ (typically) complete
  unless two or more assets have linearly dependent payoffs
- $Z > S$: more assets than states $\Rightarrow$ (typically) complete
  unless several assets have linearly dependent payoffs
  - $Z - S$ assets will be *redundant*: removing them from the economy will not impact the set of equilibria

Examples

i) Markets are complete for the following asset structures:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

i) Markets are incomplete for the following asset structures:

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$$

Sequential Trading with General Asset Structure

- Lagrangian

$$\mathcal{L} = u^i\left(c_0^i\right) + \beta^i \sum_s \pi_1(s)\, u^i\left(c_1^i(s)\right) - \lambda_0^i \left(c_0^i + \sum_z q_0^z a_0^{iz} - \bar{y}_0^i\right)$$
$$- \sum_s \lambda_1^i(s) \left(c_1^i(s) - \bar{y}_1^i(s) - \sum_z d_1^z(s)\, a_0^{iz}\right)$$

- Optimality conditions given by

$$\frac{d\mathcal{L}}{dc_0^i} = \frac{\partial u^i}{\partial c_0^i} - \lambda_0^i = 0 \quad \text{and} \quad \frac{d\mathcal{L}}{dc_1^i(s)} = \beta^i \pi(s) \frac{\partial u^i}{\partial c_1^i(s)} - \lambda_1^i(s) = 0$$

$$\frac{d\mathcal{L}}{da_0^{iz}} = -q_0^z \lambda_0^i + \sum_s \lambda_1^i(s)\, d_1^z(s) = 0 \Rightarrow \text{Euler equation}$$

- Marginal cost of purchasing a unit of asset z by individual i at date 0, given $q_0^z \lambda_0^i$ in consumption units, must be equal to the marginal benefit of doing so, given by $\sum_s \lambda_1^i(s)\, d_1^z(s)$.

Sequential Trading with General Asset Structure

- Euler equations if fundamental equation of asset pricing:

$$q_0^z \lambda_0^i = \lambda_1^i (s) \, d_1^z (s) = 0 \Rightarrow q_0^z = \sum_s \pi (s) \underbrace{\frac{\beta^i \frac{\partial u^i}{\partial c_1^i (s)}}{\frac{\partial u^i}{\partial c_0^i}}}_{= m^i (s) \text{ (SDF)}} d_1^z (s)$$

$$\Rightarrow \boxed{q_0^z = \sum_s \pi (s) \, m^i (s) \, d_1^z (s)}$$

where $m^i (s)$ denotes individual i's *stochastic discount factor* (SDF)

Equivalence

- Once-and-for-all-trading and trading with assets only equivalent if markets are complete!
- Budget constraint consolidation argument:
  - Use date-1 budget constraint to solve for

$$\boldsymbol{a}_0^i = \boldsymbol{D}^{-1}\left(\boldsymbol{c}_1^i - \bar{\boldsymbol{y}}_1^i\right)$$

    - $\boldsymbol{D}$ must be invertible $\Rightarrow$ Only possible when $S = Z$ and $\boldsymbol{D}$ is full rank (complete markets)
    - If $Z > S$, drop the assets with linearly dependent payoffs
  - Define vector of asset prices:

$$\boldsymbol{q}_0 = \left(q_0^1, \ldots, q_0^z, \ldots, q_0^Z\right)_{1\times Z}$$

  - Substitute $\boldsymbol{a}_0^i$ into date-0 budget constraint, $c_0^i + \boldsymbol{q}_0\boldsymbol{a}_0^i = \bar{y}_0^i$, so

$$c_0^i + \boldsymbol{q}_0\boldsymbol{D}^{-1}\left(\boldsymbol{c}_1^i - \bar{\boldsymbol{y}}_1^i\right) = \bar{y}_0^i \Rightarrow \boxed{c_0^i + \boldsymbol{q}_0\boldsymbol{D}^{-1}\boldsymbol{c}_1^i = \bar{y}_0^i + \boldsymbol{q}_0\boldsymbol{D}^{-1}\bar{\boldsymbol{y}}_1^i}$$

  - $\boldsymbol{q}_0\boldsymbol{D}^{-1}$ maps to $p_1(s)$ if the consolidation occurs
- If assets are Arrow-Debreu securities
  - $\boldsymbol{D}$ is identity of dimension $S = Z$ and $q_0^z = \mu_0(s)$
- Incomplete markets: not possible to consolidate $\Rightarrow$ Equivalence fails!

Spanning through Options

- Idea: derivatives complete markets
- Consider economy with a single primary asset with a payoffs

$$\boldsymbol{d}_1^1 = (4, 3, 2, 1)$$

- Options are derivative assets whose payoffs depend on the primary asset. The payoff of a call option with strike K is:

$$\max\left\{\boldsymbol{d}_1^1 - K, 0\right\}$$

- We can thus use call options with different strike prices, say $K = \{3.5, 2.5, 1.5\}$, to generate derivative securities that when combined induced full asset spanning, that is, complete markets.
  i) Call Option with $K = 3.5$: $\max\left\{\boldsymbol{d}_1^1 - 3.5, 0\right\} = (0.5, 0, 0, 0))$
  ii) Call Option with $K = 2.5$: $\max\left\{\boldsymbol{d}_1^1 - 2.5, 0\right\} = (1.5, 0.5, 0, 0)$
  iii) Call Option with $K = 1.5$: $\max\left\{\boldsymbol{d}_1^1 - 1.5, 0\right\} = (2.5, 1.5, 0.5, 0)$
- Markets are now **complete**
- Reneging on contracts also completes markets $\Rightarrow$ Bankruptcy

Static meets Dynamic/Stochastic

- Every positive and normative property studied in Block I for static exchange economies applies unchanged to complete market economies
  - Welfare theorems
  - Existence, uniqueness, convergence
  - Excess demand theorem, etc.

"The dynamic stochastic model is a special case of the static model".

- $T = \infty$ does not change these results
- Double infinite may $\Rightarrow$ OLG

Extension: No Initial Consumption

- No initial consumption $\Rightarrow$ Portfolio choice problem
- Preferences

$$\sum_s \pi(s)\, u^i\left(c^i(s)\right)$$

- Resource constraints

$$\sum_i c^i(s) = \sum_i \bar{y}^i(s)\,,\ \forall s$$

- All equilibrium notions apply assuming $c_0^i = \bar{y}_0^i = 0$
- If $S = 2 \rightarrow$ stochastic Edgeworth box economy

Extension: Multiple Goods I

- Preferences

$$V^i = u^i\left(\left\{c_0^{ij}\right\}_{j\in\mathcal{J}}\right) + \beta^i \sum_s \pi_1(s)\, u^i\left(\left\{c_1^{ij}(s)\right\}_{j\in\mathcal{J}}\right)$$

- Resource constraints

$$\sum_i c_0^{ij} = \sum_i \bar{y}_0^{ij},\ \forall j \qquad \text{and} \qquad \sum_i c_1^{ij}(s) = \sum_i \bar{y}_1^{ij}(s),\ \forall j,\ \forall s$$

- Equilibrium once-and-for-all-trading

$$\max_{\left\{c_0^i, c_1^{ij}(s)\right\}} u^i\left(\left\{c_0^{ij}\right\}_{j\in\mathcal{J}}\right) + \beta^i \sum_s \pi_1(s)\, u^i\left(\left\{c_1^{ij}(s)\right\}_{j\in\mathcal{J}}\right) \quad \text{s.t.}$$

$$\sum_j p_0^j c_0^{ij} + \sum_s \sum_j p_1^j(s)\, c_1^{ij}(s) = \sum_j p_0^j \bar{y}_0^{ij} + \sum_s \sum_j p_1^j(s)\, \bar{y}_1^{ij}(s), \quad \forall i$$

- How many budget constraints do we have?

Extension: Multiple Goods II

- Equilibrium with assets

$$\max_{\left\{c_0^i, c_1^{ij}(s), \left\{a_0^{iz}\right\}_{z\in\mathcal{Z}}\right\}} u^i\left(c_0^i\right) + \beta^i \sum_s \pi_1(s)\, u^i\left(\left\{c_1^{ij}(s)\right\}_{j\in\mathcal{J}}\right) \quad \text{s.t.}$$

$$c_0^{i1} + \sum_{j=2}^{J} p_0^j c_0^{ij} + \sum_z q_0^z a_0^{iz} = \bar{y}_0^{i1} + \sum_{j=2}^{J} p_0^j \bar{y}_0^{ij}$$

$$c_1^{i1}(s) + \sum_{j=2}^{J} p_1^j(s)\, c_1^{ij}(s) = \bar{y}_1^{i1}(s) + \sum_{j=2}^{J} p_1^j(s)\, \bar{y}_1^{ij}(s) + \sum_z d_1^z(s)\, a_0^{iz},\ \forall s$$

- Financial assets in units of good 1
- Note that $p_1^j(s)$ means something different in
  - once-and-for-all trading equilibrium (previous slide)
  - sequential equilibrium trading with assets (this slide)
- How many budget constraints do we have?

References I

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